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ANNUAL SURVEY OF GENERAL ECONOMIC THEORY: THE PROBLEM OF INDEX NUMBERS

By Ragnar Frisch

1. INTRODUCTION

The problem of how to construct an index number is as much one of economic theory as of statistical technique. Indeed, all discussions about the "best" index formula, the "most correct" weights, etc., must be vague and indeterminate so long as the meaning of the index is not exactly defined. Such a definition cannot be given on empirical grounds only but requires theoretical considerations. It seems fitting, therefore, to include a survey of this subject in the Econometrica surveys of *general economic theory.*

The index-number problem arises whenever we want a quantitative expression for a *complex* that is made up of individual measurements for which no common *physical* unit exists. The desire to unite such measurements and the fact that this cannot be done by using physical or technical principles of comparison only, constitute the essence of the index-number problem and all the difficulties center here.

Constructions of this sort may be attempted for many different kinds of measurements: prices and quantities of economic goods, traffic intensities, fertility of the soil, etc. Each kind of index-number may be considered separately or in connection with other kinds, as, for instance, when a price-index and a quantity index are used as elements in Irving Fisher's equation of exchange[1] or, more generally, when an index-number is constructed for each of several factors that contribute to a joint result, as in the case considered by Wisniewski.[2] This general aspect of the problem will not be treated here. The survey will be confined to the problem of *price* index numbers only.

Even within this narrower field several interpretations are possible. The variety of purposes is well discussed by Wesley C. Mitchell who says, amongst other things:[3]

> For example, some one should compile a special series for forecasting changes in business conditions. The compiler might select those commodities whose prices in the past have given the earliest and most regular indications of changes that subsequently occurred in the larger index numbers, he might weight these series in accordance with their past reliability as price barometers, and he might use whatever method of averaging the fluctuations gave the best results for his

[1] *The Purchasing Power of Money,* first edition, New York, 1911.

[2] *Journ. Am. Stat. Ass.,* March, 1931.

[3] *The Making and Using of Index Numbers,* U. S. Bureau of Labor Statistics, Bulletin No. 284, October 1921, page 24.

purpose. Such a series probably would not be a reliable measure of variations in the purchasing power of money, but it probably would be better adapted to its special purpose . . .

The present survey will not discuss index numbers of this sort but be confined to those whose object is to measure some sort of purchasing power. Only the fundamental theoretical problems will be considered. We shall not take up practical questions connected with the reliability or fullness of the data nor discuss questions arising out of the practical necessity of using a limited number of *representative* articles.

The main contributions in the field, as thus circumscribed, are to be found in the works of: W. Stanley Jevons,[4] F. Y. Edgeworth,[5] C. H. Walsh,[6] Irving Fisher,[7] Wesley C. Mitchell,[8] A. C. Pigou,[9] Carrado Gini,[10] François Divisia,[11] René Roy,[12] J. M. Keynes,[13] A. L. Bowley,[14] Gottfried Haberler,[15] L. V. Bortkiewicz,[16] A. A. Konüs,[17] R. G. D. Allen,[18] and Hans Staehle.[19] I may perhaps also add a reference to some of my own papers.[20]

The standard older work is Edgeworth's. Among the more recent contributions, Staehle's seems the most original and constructive, although perhaps lacking somewhat in simplicity and perspicuity.

I shall now summarize the salient features of the progress made in

[4] Papers reprinted in *Investigations in Currency and Finance*, London, 1909.

[5] Various papers, most of which are reproduced in Vol. I of *Papers Relating to Political Economy*, London, 1925.

[6] *The Measurement of General Exchange Value*, New York, 1901. Also *Quar. J. Ec.*, 1924.

[7] *Loc. cit.* and *The Making of Index Numbers*, first edition, Boston, 1922.

[8] *Loc. cit.*

[9] *Wealth and Welfare*, London, 1912, Chap. III, second edition, 1924, Chap. V.

[10] *Metron*, July, 1924, and Feb., 1931.

[11] "L'indice monetaire et la théorie de la monnaie." Separately and in *Revue d'Economie politique*, 1925.

[12] *Rèvue d'Economie politique*, 1927.

[13] *A Treatise on Money*, London, 1930, Vol. I, Book II.

[14] *Jour. Roy. Stat. Soc.*, 1919, 1921, and 1926. *Econ. Journ.*, 1923 and 1928.

[15] *Der Sinn der Indexzahlen*, Tübingen, 1927. Also *Weltw. Arch.* 1929.

[16] *Nordic Statistical Journal*, 1923, 1924, 1932.

[17] Economic Bulletin. Conjuncture-Institute of Moscow. n. 9/10, 1924 (Russian).

[18] *Economica*, May, 1933.

[19] *Archiv. f. Sozialw. u. Sozialpol.*, 1932. *International Comparisons of Food Costs*, 1934 (In studies and reports on the International Labour Office. Series N., No. 20), ECONOMETRICA, 1934, page 59, and *The Review of Economic Studies*, June, 1935.

[20] *New Methods of Measuring Marginal Utility*, Tübingen, 1932. Section 9. Also *Journ. Am. Stat. Ass.*, Dec., 1930.

this field by these authors, and on certain points shall also try to extend the analysis a little further.

2. THE ATOMISTIC APPROACH

There are two fundamentally different ways in which the problem of price index numbers may be approached. We term them the *atomistic* and the *functional* approach. In the atomistic approach the prices $p^1, p^2, \cdots, p^N$, and the quantities $q^1, q^2, \cdots, q^N$, of the various goods are considered—at least in the main—as two sets of *independent* variables. And the desideratum is to define a certain function of these $2N$ variables which will give a plausible expression for the "general movement" of prices. In the functional approach certain *characteristic relations* are assumed to exist between prices and quantities. This changes the whole nature of the problem. While in the atomistic approach a logical and unique definition of the index number is impossible, such a definition becomes, as we shall see, possible in the functional approach.

Consider first the atomistic approach. A typical example is the way in which Irving Fisher lets a mechanical balance illustrate the two sides of the equation of exchange.[21] On one side are hung at different distances from the fulcrum a loaf, a coal scuttle, and a roll of cloth, the three being kept in balance by a purse suspended on the other side. The weights represent quantities and the "arms" (distances from fulcrum) prices. Alternative magnitudes of the weights and arms are discussed, illustrating the conception of prices and quantities as independent variables. The *averaging* of the prices is illustrated by hanging all three items *in one average point.* The arm of this point—if the momentum is to be the same—is, of course, uniquely determined, thus giving the impression that the conception of an average price is well defined. This latter part of the example is dangerously misleading. Indeed, that feature of the example which entails the unique determination of the average arm is the *physical* commensurability of the weights, one pound of bread being—from the viewpoint of the mechanical balance—equal to one pound of coal, etc. But it is precisely the *absence* of this physical commensurability that constitutes the index-number problem.

The indeterminateness created by physical incommensurability is, in fact, very troublesome in the atomistic approach. Various attempts have been made to overcome it.

First we have what Edgeworth called the *indefinite standard* approach,[22] which may more appropriately be called the *stochastic*

[21] *The Purchasing Power of Money,* Chap. II, Section 3.
[22] *Papers,* Vol. I, pages 196 and 235.

approach. Here the assumption is made that any change that takes place in the "price level" *ought*, so to speak, to manifest itself as a proportional change of all prices. Whatever deviation there is from this strict proportionality must be looked upon as due to *other causes* than those we think of when we speak of the price level change. How *in concreto* these two sets of causes are distinguished is not essential for the conceptual existence of a proportionality factor, but as a matter of fact the distinction is usually—more or less explicitly—taken to be that between monetary and non-monetary causes.

According to this conception, the deviation of the individual price changes from proportionality must be considered more or less as errors of observation. But then the application of the theory of errors should enable us to determine the underlying proportionality factor. If we compare the two price situations, 0 and 1 (denoted by subscripts), any of the individual price ratios p_1^k/p_0^k $(k=1, 2, \cdots, N)$ may in a first approximation be taken as just as good an estimate of the price level change as any other of these ratios. Consequently, their simple average will give an estimate of the price level change between 0 and 1. If weights are to be applied at all in this averaging, they should express the *precisions* of the individual observations; these need not be proportional to the economic importance of the goods, as measured, for instance, by the *quantities*, q^k, or the *values*,[23] p^kq^k. Refining this type of analysis, one is led to study the *statistical distribution* of the individual ratios p_1^k/p_0^k. Criteria may be developed for testing the normality of the distribution, the independence of the observations, and so on. Considerations of this sort lead to the adoption of averages different from the arithmetic, in particular—in the case of skew distributions—the geometric average,[24] and—if the observations are noticeably dependent—the weighted median[25] (Laplace's "Method of Situation").

Thus, the notion of a "price level" here becomes essentially *stochastic*. We may make probability statements about it, but not "exact" statements like those we make about other magnitudes in an economic theoretical scheme. In consequence, the price level has a meaning only when a great number of individual goods enter into it. As Edgeworth says:[26] "To me the conception appears somewhat indefinite as applied to two or a few articles and without relation to the theory of averages."

This conclusively rules out the possibility of using the above concept as the *definition* of the price level. At least for many kinds of economic analyses it will be an impossible concept, for instance, in

[23] Edgeworth, Vol. 1, page 243. "Assuming . . . accidental deviations. . . ."
[24] Edgeworth, Vol. I, page 238.
[25] Edgeworth, Vol. I, page 249.
[26] *Econ. Journal*, 1923, page 343.

cases where it is desired to build up a hierarchical order of indices, each index being itself a composite of lower order indices. Furthermore, the logical basis of the whole concept seems untenable. We cannot assume that the "monetary factor" will manifest itself as a proportional change of all prices. I am, therefore, in agreement with Keynes when he vigorously criticizes the stochastic definition of the price level as being "root-and-branch erroneous,"[27] and with Gini, who says, "qu'on ne peut arriver à résoudre le problème de la manière envisagée."[28]

I am here speaking of the exact *definition* of the price level concept. The study of price ratio distributions and similar questions is in itself highly significant from other different viewpoints, for instance, as a means of describing concretely the various goods according to their price behavior, etc., as done by Fredrick C. Mills[29] and others, or as a means of elucidating the nature of various index number formulae that have been suggested heuristically as approximations.

Another attempt to escape indeterminateness—while still employing the atomistic viewpoint—is the *test-approach*. It consists in formulating certain formal tests regarding the function that expresses the price level change from one situation to another. The exponent of this approach is Irving Fisher. Let P_{01} be the index number that expresses the ratio between the price level in point 1, the "object point," and the price level in point 0, the "base point." P_{01} is assumed to depend on the prices $p_0^1 \cdots p_0^N$, $p_1^1 \cdots p_1^N$, and the quantities $q_0^1 \cdots q_0^N$, $q_1^1 \cdots q_1^N$. Some of the more important tests are:

The Identity test: $P_{00}=1$.

The Point reversal test ("time" reversal test): $P_{01}P_{10}=1$.

The Circular test: $P_{01}P_{12}=P_{02}$.

The Commensurability test: P_{01} shall not change by changing the unit of measurement for any of the individual goods.

The Determinateness test: P_{01} shall not become zero, infinite, or indeterminate, if an individual price or quantity becomes zero.

The Proportionality test: If all individual prices have changed in the same proportion from 0 to 1, P_{01} shall be equal to the common factor of proportionality.

Sauerbeck's index, i.e., the simple arithmetic mean of the price ratios,

$$P_{01}^{\text{Sau}} = \frac{1}{N}\sum \frac{p_1}{p_0} \tag{2.1}$$

[27] *A Treatise on Money*, Vol. I page 85.

[28] *Metron*, 1924, page 21.

[29] *The Behavior of Prices*, New York, 1927.

(which may be looked upon as the result obtained by the simplest possible form of the stochastic approach), satisfies only the identity, commensurability, and proportionality tests. The well known formulae of Laspeyre and Paasche,

$$(2.2) \qquad P_{01}^{\mathrm{La}} = \frac{\sum p_1 q_0}{\sum p_0 q_0},$$

$$(2.3) \qquad P_{01}^{\mathrm{Pa}} = \frac{\sum p_1 q_1}{\sum p_0 q_1},$$

satisfy the commensurability, determinateness, and proportionality tests, but not the point reversal test, and, *a fortiori*, not the circular test. The crossing between them,

$$(2.4) \qquad P_{01}^{\mathrm{Ide}} = \sqrt{P_{01}^{\mathrm{La}} \cdot P_{01}^{\mathrm{Pa}}},$$

considered by Bowley, recommended by Walsh and Pigou, and called by Fisher the "ideal" formula, satisfies the point reversal test but not the circular test. The same is true of Edgeworth's formula,

$$(2.5) \qquad P_{01}^{\mathrm{Ed}} = \frac{\sum p_1(q_0 + q_1)}{\sum p_0(q_0 + q_1)}.$$

On the other hand, the arithmetic average with constant weights,

$$(2.6) \qquad P_{01}^{\mathrm{Ari.c.w.}} = \frac{\sum p_1 q}{\sum p_0 q} \text{ (the } q\text{'s independent of the point 0 and 1)},$$

as well as the geometric average with constant weights,

$$(2.7) \qquad P_{01}^{\mathrm{Ge.c.w.}} = \frac{\Pi\, p_1^{\alpha}}{\Pi\, p_0^{\alpha}} = \frac{(p_1^1)^{\alpha^1} \cdots (p_1^N)^{\alpha^N}}{(p_0^1)^{\alpha^1} \cdots (p_0^N)^{\alpha^N}}, \text{ } \left(\textstyle\sum \alpha = 1 \text{ and the } \alpha\text{'s independent of the points 0 and 1}\right)$$

satisfy the circular test (for any set of three points for which the q's or the α's are the same). In addition, (2.6) will satisfy the other tests mentioned. For any comparison where the quantities, q, can be assumed sensibly constant, (2.6) gives, therefore, a satisfactory solution. This is, however, only a trivial case. The fundamental difficulty is that, in most cases, particularly for geographical comparisons or comparisons between remote points of time, it is absurd to assume constant q's. In any such case, we must let the formula depend on the actual q_0's and q_1's, which brings us back to formulae of types (2.1)–(2.5).

The difficulties here discussed are unavoidable so long as we maintain the atomistic viewpoint and consider the p's and q's as inde-

pendent variables. On this assumption (and assuming certain continuity properties of the index-number formula), I have indeed proved that three such fundamental tests as the commensurability, determinateness, and circular tests cannot be satisfied at the same time.[30] And, even if some of the tests are abandoned (Fisher is, for instance, willing to give up the circular test), the remaining ones do not lead to a unique formula.

Although the test approach cannot lead to one particular formula that may be taken as *the* definition of the price level, it is, however, a convenient tool for judging the comparative merits of various formulae that suggest themselves heuristically as approximations to a price level defined by some other means.

A special tool that deserves mention in connection with the atomistic approach is the *chain method*, originally introduced by Alfred Marshall.[31] This method is adapted to data where the points are ordered in a unique *sequence*, which, in practice, means time series, but not geographical data. Let P_{01} be any index formula for the direct comparison between two points, for instance, one of the formulae defined by (2.1)–(2.7). The *chain* index, $\overline{P}_{st}$, between any two points, s and t, is then defined by

$$\overline{P}_{st} = \frac{P_{01}P_{12} \cdots P_{t-1,t}}{P_{01}P_{12} \cdots P_{s-1,s}}, \tag{2.8}$$

0 being the first point existing in the data. The definition (2.8) obviously applies as well for $s<t$ as for $s \geqq t$. If $s<t$, (2.8) reduces to

$$\overline{P}_{st} = P_{s,s+1}P_{s+1,s+2} \cdots P_{t-1,t}. \tag{2.9}$$

Any chain index, $\overline{P}_{st}$, satisfies the point reversal test and the circular test no matter on what kind of elementary index it is built. If the elementary inαex meets the circular test, there is no difference between the chain index and the direct index computed by the same formula.

The chain method has been given an elegant logical justification by Divisia. His derivation is in essence as follows. The problem is to split the total value $\sum pq$ into a product of two factors,

$$PQ = \sum pq, \tag{2.10}$$

of which the first P may be taken as representing the "general level of prices" and the second Q as "the total physical volume." In order to do so, Divisia considers the historical path in the $2N$ dimensional

[30] *Journ. Am. Stat. Ass.*, Dec., 1930.
[31] *Contemporary Review*, March, 1887.

space whose coordinates are $p^1 \cdots p^N$, $q^1 \cdots q^N$. Along the historical path we have

$$PdQ + QdP = \sum pdq + qdp, \tag{2.11}$$

and, dividing this by (2.10), we get

$$d\log P + d\log Q = \sum \alpha d\log p + \sum \alpha d\log q, \tag{2.12}$$

where

$$\alpha^1 = \frac{p^1 q^1}{\sum pq} \cdots \alpha^N = \frac{p^N q^N}{\sum pq}. \tag{2.13}$$

The formula (2.12) holds good whatever the definition of P and Q, provided only that (2.10) is fulfilled. The formal analogy between the terms on the two sides of (2.12) very naturally suggests accomplishing the definition of P and Q by equating the terms of (2.12) *separately*, i.e., by putting

$$d\log P = \sum \alpha d\log p, \tag{2.14}$$

$$d\log Q = \sum \alpha d\log q. \tag{2.15}$$

The equality (2.14) is a differential definition of the price index and (2.15) a similar definition of the quantity index. If (2.14) is integrated numerically, we are merely led to a chain index of the form (2.9), or, more generally, (2.8). As the elementary formula of the chaining, we may get Laspeyre's or Passche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration. If the weights (2.13) are constant over the integration period, the result will simply be—as pointed out by Roy[32]—the geometric average (2.7). Since (2.7) satisfies the circular test, there is in this special case no difference between the chain index and the direct index.

The divergency which exists between a chain index and the corresponding direct index (when the latter does not satisfy the circular test) will often take the form of a systematic drifting. This means that, with increasing t, the ratio $\overline{P}_{st}/P_{st}(t>s)$ departs more and more from unity (upwards or downwards as the case may be). To understand this, consider the *triangle divergency*

$$D_{rst} = P_{rs}P_{st}/P_{rt}. \tag{2.16}$$

In terms of D, we have

[32] *Revue d'Économie politique*, 1927.

$$\frac{\overline{P}_{0t}}{P_{0t}} = D_{012}D_{023}D_{034} \cdots D_{0,t-1,t}. \tag{2.17}$$

If the circular test holds, $D \equiv 1$. Otherwise, it may be highly probable that D is, say, *larger* than 1. As an example, consider Sauerbeck's index. Here

$$D^{\text{Sau}}_{0,s+1,s+2} = \frac{\dfrac{1}{N}\sum \dfrac{p_{s+1}}{p_0} \cdot \dfrac{1}{N}\sum \dfrac{p_{s+2}}{p_{s+1}}}{\dfrac{1}{N}\sum \dfrac{p_{s+1}}{p_0} \cdot \dfrac{p_{s+2}}{p_{s+1}}}. \tag{2.18}$$

Formula (2.18) can be transformed by means of

$$\frac{1}{N}\sum xy = \bar{x} \cdot \bar{y} + \sigma_x \sigma_y r_{xy}, \tag{2.19}$$

where x and y are any two variables, $\bar{x}$ and $\bar{y}$ their means over N values, σ_x and σ_y their standard deviations, and r_{xy} their correlation coefficient. The relation (2.19) is verified simply by writing the formula for the correlation coefficient and rearranging the terms. (The formula also holds if all summations are taken with weights.) Equation (2.19) shows that the average of a product is larger or smaller than the product of the averages according as the two variables are positively or negatively correlated. Putting in (2.18)

$$x = \frac{p_{s+1}}{p_0}, \qquad y = \frac{p_{s+2}}{p_{s+1}},$$

we get

$$D^{\text{Sau}}_{0,s+1,s+2} = \frac{1}{1 + uv \cdot r_{xy}}, \tag{2.20}$$

where $u = \sigma_x/\bar{x}$, $v = \sigma_y/\bar{y}$ are essentially positive. On the whole, those prices that have changed *less* than the average from 0 to $s+1$, will change *more* than the average from $s+1$ to $s+2$; they will "catch up"; hence r_{xy} negative and (2.20) larger than 1. Sauerbeck's index used for chaining will, therefore, drift upwards. A similar argument shows that Laspeyres' index will drift upwards, Paasche's downwards. Even in the crossing, (2.4), some downward drifting is left, as shown experimentally by Persons.[33] The word "drifting" must not be taken to mean that the direct index is right and the chain wrong. This cannot be decided by the above formal considerations.

The chain method has been generalized by Gini into a *net-work*

[33] *Review of Economic Statistics*, May, 1921.

method,[34] applicable whether the data are ordered in a sequence or not. He proposes two formulae which may be termed Gini's aggregate crossing and two point crossing, respectively.

$$(2.21)\ P_{01}^{\mathrm{Gi.a.c.}} = \sqrt[M]{\prod_r \frac{\sum p_1 q_r}{\sum p_0 q_r}} \qquad (2.22)\ P_{01}^{\mathrm{Gi.t.p.c.}} = \sqrt{\prod_r P_{0r} P_{r1}}.$$

Here $\prod_r$ designates a product over all the M points occurring in the material. The quantity P_{0r} in (2.22) is any elementary index. Both (2.21) and (2.22) satisfy the circular test over the range to which the crossing is extended. For $M=2$, (2.21) reduces to Fisher's "ideal" formula. The inconvenience of the Gini crossings is that recomputations have to be made when more data are included. This will not occur frequently in geographical price comparisons, however, for which this method is primarily intended.

3. THE FUNCTIONAL APPROACH

In the functional approach, prices and quantities are looked upon as connected by certain typical—in point of principle, *observable*—relations. Here we do not—as in the stochastic approach—make the assumption that ideally the individual prices ought to change in the *same* proportion as we pass from one situation to another. We face the deviations from proportionality and take them merely as expressions for those *systematic* relations that serve to give an economic meaning to the index number. The resulting index will, in point of principle, appear as observable with the same sort of precision as the price of an individual commodity, provided the necessary data are available.

These data include something more than just a set of prices and a set of quantities associated with each situation, and in practice, of course, the complete data may be difficult to get. This leads to methods of approximations and limits in which one uses to a considerable extent formulae of the kind discussed in Section 2. But there is the fundamental difference that we now know the question to which an answer is sought and have, therefore, a basis for our judgment about the various formulae.

There are various alternative sets of data, each of which is *sufficient* for the functional definition of the index number. Subsequently, some of them will be mentioned. To start with, we shall indicate certain general properties which the data must have in order to make the definition possible.

Consider any two situations, 0 and 1. In the most general formulation of the problem, these situations may differ in any number of re-

[34] *Metron*, Aug., 1931, page 10.

spects: different kinds of populations, different kinds of goods transacted or consumed or produced, etc. We assume that total money expenditure is well defined and quantitatively observable in each of the situations; let it be ρ_0 and ρ_1 respectively. It must, if the concepts of prices p_t and quantities q_t are defined, be equal to

$$\rho_t = \sum p_t q_t. \tag{3.1}$$

If each of the situations, 0 and 1, is characterized by a *given set* of prices and quantities, then ρ_0 and ρ_1 will by (3.1) be two *given numbers*. In the functional approach they must not be looked upon as such, but as capable of a certain *variation*, i.e., the expenditure within situation 0 may assume different values, and similarly for 1.

Suppose that we dispose of some criterion by which it is possible to ascertain objectively whether or not a person who in 0 spends an amount ρ_0, is just "as well off" as a person who in 1 spends an amount ρ_1. If they are equally "well off," the two amounts may be called *equivalent*. In symbols, ρ_0 equiv. ρ_1. If such a criterion exists, we take the ratio

$$\overset{\text{Func}}{P_{01}} = \frac{\rho_1}{\rho_0} \quad (\text{when } \rho_0 \text{ equiv. } \rho_1), \tag{3.2}$$

as the definition of the functional price index number between 0 and 1. So far the definition is, of course, only formal; its practical value will depend on the possibility of actually finding an objective equivalence criterion. Before turning to this, note that the idea of being equally "well off" enters in some form or another as an essential element into the theories of writers on index-numbers. For instance, Knut Wicksell[35] says: ". . . müsste man zu diesem Zwecke die verbrauchten Warengattungen selbst und ihre relative Bedeutung für die wirtschaftenden Individuen anstellen . . ." Konüs takes "Konstanter Bedürfnisstand" as the criterion[36] by which to define the true index. Gini[37] presents an argument which is tantamount to defining equal "well being" by the equality of the "average level" of the marginal utilities of the individual goods. Bowley[38] defines the cost-of-living index-number by asking: "What change in expenditure is necessary, after a change of

[35] *Geldzins u. Güterpreise*, Jena, 1898, page 12. Staehle (*Intern. Comp. of Food Costs*, page 4) takes Wicksell as one of the authors who has *given up* the attempt at defining a "true" index number. This is not correct, it seems. Wicksell realized the impossibility of doing it on an atomistic basis, but at the same time saw the functional possibility, as indicated by the above quotation.

[36] Quoted after Bortkiewicz, *Nordisk Statistisk Tidskrift*, Bd. 11, page 18.

[37] *Metron*, 1924, Vol. IV. See in particular pages 16, 22, 140, and 144.

[38] *Economic Journal*, 1928, page 223.

prices, to obtain the same satisfaction as before?" and Bortkiewicz[39] requires that:

... der dem Arbeiter im Zeitraum 2 zuzubilligende Geldlohn ... ihm die gleiche Gesamtbefriedigunf sichert, wie der Geldlohn, der ihm im Zeitraum 1 zustland, oder anders angedrückt, dass der Reallohn ... gleich hoch bleibt.

Royal Meeker[40] rallies to Bowley's view. Keynes[41] says:

Two collections of commodities are equivalent if they represent ... the things which are purchased by ... two persons of equal sensitiveness and possessed of equal real incomes of utility.

Haberler[42] takes a similar position. Allen[43] and Staehle[44] define equivalence of expenditure by the fact that the two quantity combinations considered lie on the same indifference locus in a given indifference map.

A great number of other authors could also be quoted who more or less explicitly adopt the definition (3.2). This definition seems, indeed, to be the only plausible one. It is applicable not only to cost of living indices, but equally well to general indices of deferred payments, wholesale prices, etc.

How then can we obtain objective criteria for being equally "well off"? This requires, in the first place, that we segregate a certain group of individuals, the *definitional group* for the index number in question, for instance, working men's families in the case of a cost of living index, wholesale merchants (or perhaps retailers?) in the case of an index of wholesale prices, etc. We assume that the question of the definitional group is settled. In the second place, it must be possible to observe objectively one or more parameters, $\mu, \nu, \cdots, \lambda$, that characterize the behavior of a *typical individual* in the definitional group, and which may be taken as indicators of equal "well being." Let us call them the *behavioristic* parameters. Suppose that it is possible to observe objectively—within each of the situations considered—the *covariation* between the money expenditure and the behavioristic parameters. Let this function for the situation t be

$$(3.3) \qquad \rho_t = \rho_t(\mu, \nu, \cdots, \lambda).$$

The functional price index between 0 and 1—which now may be called a general parametric index—will then be

[39] *Nordic Statistical Journ.*, 1932, page 17.
[40] *Encyclopedia Britannica*, article on "Cost of Living."
[41] *A Treatise on Money*, Vol. I, page 97.
[42] *Der Sinn der Indexzahlen*, 1927, pages 77–83.
[43] *Economica*, May, 1933.
[44] *Review of Economic Studies*, June, 1935.

$$(3.4) \qquad P_{01}^{\text{Par}} = \frac{\rho_1(\mu, \nu, \cdots, \lambda)}{\rho_0(\mu, \nu, \cdots, \lambda)}.$$

In general P_{01}^{Par} will depend on $\mu, \nu, \cdots, \lambda$. But, conceivably, the functions ρ_1 and ρ_0 may be such that (3.4) is independent of these parameters and depends only on the subscripts 0 and 1. If this is so for any situations 0 and 1, we shall say that the index satisfies the *expenditure proportionality* condition. In this case, a small 0 expenditure and a large 0 expenditure must be multiplied by the same number in order to get the equivalent 1 expenditure.

Formula (3.4) fulfils the circular test identically in $\mu, \nu, \cdots, \lambda$, and whatever the functions ρ_t (provided only they are single-valued). Thus, by adopting an appropriate theoretical basis, we satisfy a desire which is ineradicable on the part of most practical statisticians but which it is virtually impossible to satisfy with the usual formulae of the atomistic type. In this survey we shall consider various methods of constructing behavioristic parameters.

4. THE INDIFFERENCE METHOD

One method is to start from the concept of an indifference map and a choice-indicator for the typical individual of the definitional group. The indifference map is the family of indifference loci or indifference surfaces through the N dimensional quantity space $q^1 \cdots q^N$. They are defined in the usual Edgeworth-Fisher-Pareto way. An indifference function is any function of $q^1 \cdots q^N$ that is constant along any indifference locus, in other words, that has the indifference loci as its contour surfaces. An indifference function which has the further property of increasing monotonically as we pass from one indifference locus to another that is preferred to the first, is a choice-indicator or, shorter, an indicator.

If

$$(4.1) \qquad I = I(q^1 \cdots q^N), \text{ or, shorter, } I = I(q),$$

is an indicator, any function $\bar{I}(q)$ obtained by a monotonically increasing transformation, i.e., by a transformation

$$(4.2) \qquad \bar{I} = F(I),$$

where F is a monotonically increasing function of one variable, will also be an indicator. The derivatives of I we denote

$$I^h = \frac{\partial I}{\partial q^h}, \qquad I^{hk} = \frac{\partial^2 I}{\partial q^h \partial q^k}.$$

The vector whose components are $I^1 \cdots I^N$ indicates the *preference direction* (the normal of the indifference surface).

Assuming that there exists an indicator, $I(q)$, for the definitional group in question implies that the typical individual has the same tastes (but not necessarily equal resources) in the various situations considered. This restrictive assumption can, of course, only be made for comparison between places that are not too different or points of time not too remote.

Further, we must make an assumption about the strategic behavior of the typical individual. We shall assume that he tries to maximize $I(q)$ on the assumption that his total expenditure is given and he is confronted with certain price functions (from his viewpoint they will be supply functions):

$$p^h = \pi_t^h(q^1 \cdots q^N) \text{ or, shorter, } p = \pi_t(q), \tag{4.3}$$

with flexibilities

$$\check{\pi}_t^{hk}(q^1 \cdots q^N) = \frac{\partial \pi_t^h}{\partial q^k} \cdot \frac{q^k}{\pi_t^h}. \tag{4.4}$$

Strictly speaking, we should not assume that his total expenditure is given, but only that it is bounded above. In practice the two assumptions will amount to the same, except in extraordinary cases where consumption has passed the saturation point.

When the expenditure ρ_t is given, the *budget surface* is

$$\begin{gathered}\pi_t^1(q^1 \cdots q^N) \cdot q^1 + \cdots + \pi_t^N(q^1 \cdots q^N) \cdot q^N = \rho_t, \\ \text{or, shorter, } \sum \pi_t q = \rho_t.\end{gathered} \tag{4.5}$$

If (4.3) is independent of the q's (but depending of course on t and h), (4.5) is a plane. The equilibrium point in t is

$$q_t = \text{a point in the surface (4.5) that maximizes } I(q). \tag{4.6}$$

The corresponding equilibrium price situation is

$$p_t = \pi_t(q_t). \tag{4.7}$$

(4.6) is the general definition of the equilibrium point. If the partial derivatives exist and are continuous, it is the solution of (4.5) and the tangency conditions

$$\frac{I^1}{\dfrac{\partial \rho_t}{\partial q^1}} = \frac{I^2}{\dfrac{\partial \rho_t}{\partial q^2}} = \cdots = \frac{I^N}{\dfrac{\partial \rho_t}{\partial q^N}}. \tag{4.8}$$

(4.8) expresses the generalized "Gossens law," namely, that the

marginal utilities" are proportional to the "marginal expendivities." The general expression for the latter is

(4.9) $$\frac{\partial \rho_t}{\partial q^h} = \pi_t^h\left(1 + \frac{1}{\pi_t^h q^h} \sum_k \check{\pi}_t^h q^k \cdot \pi_t^{kh}\right).$$

If π_t^h depends only on q^h, (4.9) reduces to

(4.10) $$\frac{\partial \rho_t}{\partial q^h} = \pi_t^h(1 + \check{\pi}_t^h) \text{ where (4.11) } \check{\pi}_t^h = \check{\pi}_t^{hh}.$$

If the prices are constant, (4.10) further reduces to

(4.12) $$\frac{\partial \rho_t}{\partial q^h} = p_t^h.$$

In the case (4.12), there passes one, and only one, budget manifold—now a plane—through each q. Indeed, its normal (the price vector p) must by (4.8) go in the preference direction through q, which determines the plane uniquely (a proportional change in all prices and in money expenditure will, of course, give the same plane). In this case we may unambiguously speak of "the budget plane through q." In the general case, different budget manifolds may have the same normal in q. We may now speak of "the t budget surface through q."

The equilibrium quantities q_t are functions of ρ,

(4.13) $$q_t^h = E_t^h(\rho) \text{ or, shorter, } q_t = E_t(\rho).$$

They are the *Engel functions* for t. They describe how—in t—the consumption of the various commodities changes with total expenditure. Each such function may be represented as a one dimensional *Engel-curve.* The complete set of Engel functions (4.13) define a one dimensional path—the *expenditure expansion path,* or, shorter, the expansion path—in the N-dimensional quantity space. Each situation t has its path. If (4.3) is independent of q, the prices are *path constant,* i.e., independent of the path. Figure 1 represents such a case for $N=2$.

With each point on a given path is associated a value of ρ and a value of I. We make the fundamental

(4.14) *Monotonicity assumption.* Along any expansion path ρ and I always change in the same direction.

The function

(4.15) $$\rho = \rho_t(I)$$

is single valued if (4.14) holds.

In terms of (4.15) the functional index—now called an indifference-defined index—will be

$$P_{01}^{\mathrm{Ind}} = \frac{\rho_1(\mathrm{I})}{\rho_0(\mathrm{I})}. \tag{4.16}$$

(4.16) is, of course, a special case of (3.4), and, consequently, satisfies the circular test.

It also satisfies the proportionality test. Indeed, if $\pi_1{}^h = c\pi_0{}^h$, c being a constant independent of h, we have $\check{\pi}_1{}^{hk} = \check{\pi}_0{}^{hk}$ and, hence, $\partial\rho_1/\partial q^h = c\cdot\partial\rho_0/\partial q^h$. Consequently, if ρ_0 is any 0 expenditure, the 1 ex-

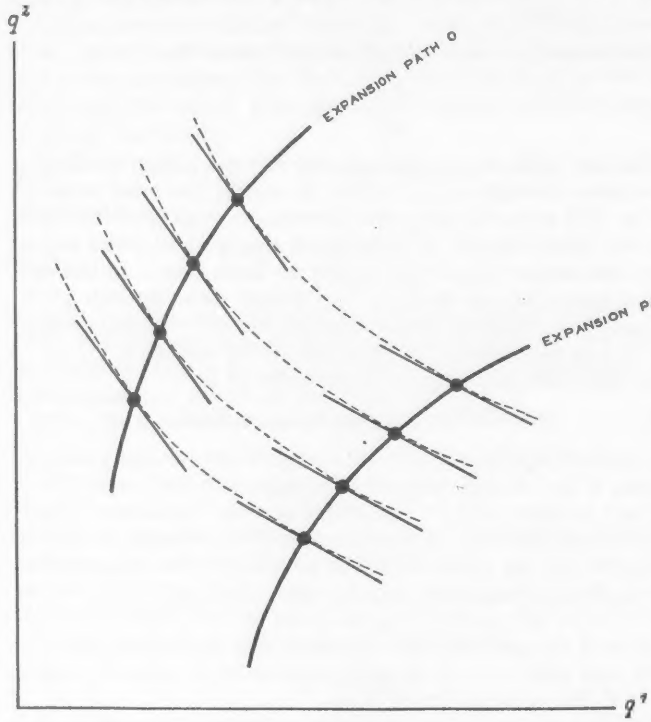

FIGURE 1

penditure, $\rho_1 = c\cdot\rho_0$, will give the same equilibrium point, i.e., $q_1 = q_0$, so that $P^{\mathrm{Ind}}{}_{01} \equiv c$ for any ρ_0, i.e., for any I. This argument shows that any two situations with proportional prices or price functions have the same expansion path, and with proportional expenditure variations along the path.

(4.16) also satisfies most other tests which it is plausible to advance.

In the general case, (4.16) will depend on I. If it does not, we have expenditure proportionality.

5. THE THEORY OF LIMITS

If an indicator, $I(q)$, is given, and each situation characterized by a set of prices or by a set of price functions, a perfectly definite price index can—as shown in Section 4—be constructed between any two situations. We now turn to the question of how the index thus defined can be approximated by using a less complete set of data. We shall first consider *limits*. To clarify the situation we must discuss also certain arguments that do not give limits but are generally believed to do so.

Pigou, Keynes, Gini, Konüs, Bortkiewicz, Bowley, Allen, and Staehle, have considered the question. Quite a bit of confusion exists regarding the assumptions made and propositions formulated by these authors. I shall attempt to state this exactly and also to give proofs. As a rule, the above authors assume—explicitly or tacitly—locally constant prices (linear budget manifolds). The following exposition shows that this is to a large extent unnecessary.

$$R_0(q) = \sum \pi_0(q) \cdot q, \tag{5.1}$$

where the π_0 are defined by (4.3), is the total money expenditure involved in buying q in the 0 situation. For brevity: $R_0(q)$ is the value of

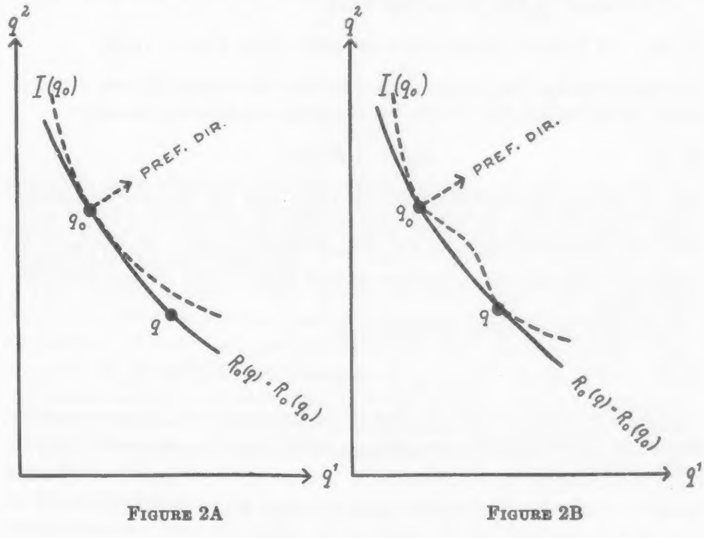

FIGURE 2A FIGURE 2B

q in 0, or the 0 value of q. This definition applies whether prices are path constant or not.

It is particularly interesting to compare an arbitrarily given q with a q_0 *that lies on the expansion path* for the 0 situation (i.e., that will for a certain money expenditure be the equilibrium complex under 0 price functions), for brevity, q_0 is 0-adapted. This comparison leads to

The general adaptation proposition. Any complex, q, that has the same 0-value as a certain 0-adapted complex, q_0, can at most give the same satisfaction as q_0, i.e.,

$$\text{(5.2)} \qquad \text{If } R_0(q) = R_0(q_0),\ q_0 \text{ 0–adapted, then } \mathrm{I}(q) \leqq \mathrm{I}(q_0).$$

This simply follows from the equilibrium definition. Indeed, if $R_0(q) = R_0(q_0)$ and $I(q) > I(q_0)$, q_0 would not satisfy (4.6). Figures 2A and 2B are examples in two goods.

The argument shows that the equality sign to the right in (5.2) only holds if $q = q_0$ or the equilibrium adaptation is non-unique (as in Figure 2b). We therefore also have the proposition

$$\text{(5.3)} \qquad \left.\begin{array}{r} \text{If } R_0(q) = R_0(q_0) \\ q_0 \text{ 0–adapted} \\ q \text{ non–equal } q_0 \\ \text{adaptation unique} \end{array}\right\} \text{then } I(q) < I\acute{c}\ (q_0).$$

If we admit (4.14), we further have

$$\text{(5.4)} \qquad \text{if } R_0(q) < R_0(q_0),\ q_0 \text{ 0–adapted, then } I(q) < I(q_0).$$

Indeed, consider the 0-expenditure surface through q (Figure 3 represents the situation for $N = 2$). Its running coordinates $\bar{q}$ satisfy

$$\text{(5.5)} \qquad R_0(\bar{q}) = R_0(q).$$

Let $\bar{q}_0$ be the intersection between (5.5) and the 0-expansion path. Then $R_0(q) = R_0(\bar{q}_0)$, $\bar{q}_0$ 0-adapted, so that by (5.2) $I(q) \leqq I(\bar{q}_0)$. But, since by hypothesis $R_0(q) < R_0(q_0)$ and, consequently, $R_0(\bar{q}_0) < R_0(q_0)$, we have by the monotonicity assumption $I(\bar{q}_0) < I(q_0)$, so that *a fortiori* $I(q) < I(q_0)$.

(5.2) and (5.4) can be summarized.

$$\text{(5.6)} \qquad \text{If } R_0(q) \leqq R_0(q_0),\ q_0 \text{ 0–adapted, then } I(q) \leqq I(q_0).$$

Inversely, any complex, q, that gives the same or a higher satisfaction than the 0-adapted complex, q_0, must have the same or a higher 0-value than q_0, i.e.,

$$\text{(5.7)} \qquad \text{if } 0(q) \geqq I(q_0),\ q_0 \text{ 0–adapted, then } R_0(q) \geqq R_0(q_0).$$

Indeed, if $R_0(q) < R_0(q_0)$, we would by (5.4) have $I(q) < I(q_0)$.

But from $I(q) \leqq I(q_0)$, we cannot conclude $R_0(q) \leqq R_0(q_0)$, i.e., a complex that is more 0-expensive than a certain 0-adapted complex may give less satisfaction, for instance, the point q in Figure 3.

The above propositions contain only one complex that is equilibrium adapted, i.e., only for one of the complexes do we consider a corre-

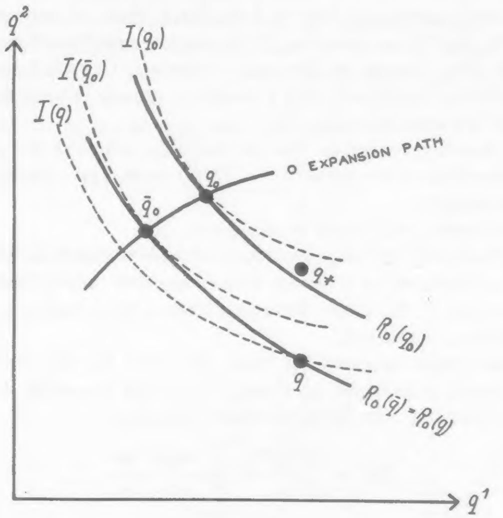

FIGURE 3

sponding price situation. Later we shall consider two price situations.

(5.2)–(5.6), where the premise concerns the expenditure, are easily remembered because the highest satisfaction always pertains to the equilibrium adapted situation.

The axiomatic basis of the above propositions is only (4.6) and (4.14), the curves in Figures 2 and 3 merely serving as illustrations, not as proofs. If it is desired, part of the conclusions can be related to the *convexity* of the indifference surfaces.

We now turn to a consideration of certain special limits and criteria.

The Pigou criteria. Pigou defines a change in the National Dividend between two periods as the ratio of the money incomes divided by the price level change, in other words, as the ratio of the *deflated* incomes. In *Wealth and Welfare*, he considers in particular the case of a constant money income, thus reducing the problem to the analysis of a price

index. This procedure, he says, is only "for simplicity of discussion—no difference in substance is made—."[45] This is hardly correct, because the whole problem of *expenditure proportionality* is thus ignored. But, waiving this for a moment, let us consider his reasoning about the price index. He wants to construct a barometer that always changes with the price situation in the opposite direction to the total satisfaction obtained by a person with a constant money expenditure. He does not find any such barometer. But he formulates what we may call

(5.8) *The first Pigou criterion.* If Laspeyre's and Paasche's formulae indicate a price change in the same direction, the total satisfaction obtained by an individual with a constant money income must have changed in the opposite direction.

Pigou, therefore, more or less heuristically adopted the geometric mean between these two indices (i.e. (2.4)) as an *approximation* to the barometer sought.

(5.8) is correct, but it can be sharpened into

Proposition (5.9). If Laspeyre's formula indicates a price decline, the satisfaction obtained by a person with a constant money income must have increased. If Paasche's formula shows a price increase, satisfaction must have decreased.

Moreover, these propositions hold, not only for the case of path constant prices considered by Pigou, but holds generally, if we only adopt the following slightly generalized definitions

$$P_{01}^{\mathrm{La}} = \frac{R_1(q_0)}{R_0(q_0)} = \frac{\sum \pi_1(q_0)\cdot q_0}{\sum \pi_0(q_0)\cdot q_0}, \tag{5.10}$$

$$P_{01}^{\mathrm{Pa}} = \frac{R_1(q_1)}{R_0(q_1)} = \frac{\sum \pi_1(q_1)\cdot q_1}{\sum \pi_0(q_1)\cdot q_1}. \tag{5.11}$$

For path constant prices, of course, (5.10) and (5.11) reduce to (2.2) and (2.3).

(5.9) is proved thus: If $R_1(q_1) = (R_0(q_0)$ and $R_0(q_0) > R_1(q_0)$ so that $R_1(q_0) < R_1(q_1)$, we have, by (5.4), $I(q_0) < I(q_1)$. Similarly for the latter part of (5.9).

Later—in the *Economics of Welfare*—Pigou considered the satisfaction barometer *directly* without reasoning via a constant expenditure. This analysis is more satisfactory because it does not assume expenditure-proportionality. Pigou wants a direct barometer that moves in the same direction as total satisfaction and considers for this purpose the ratios obtained by deflating the relative money income by Laspeyre's and Paasche's price indices respectively,[46] i.e.,

[45] *Wealth and Welfare*, 1912, page 34.

[46] *Econ. of Welfare*, second edition, 1924, page 54.

$$(5.12) \qquad \frac{\rho_1}{\rho_0}\cdot\frac{1}{P^{\mathrm{La}}_{01}} \text{ and } \frac{\rho_1}{\rho_0}\cdot\frac{1}{P^{\mathrm{Pa}}_{01}}.$$

He formulates what may be called

(5.13) *The second Pigou criterion.* If both ratios (5.12) are larger (smaller) than unity, total satisfaction must have increased (decreased) when passing from 0 to 1.

The proposition is correct but may be sharpened into:

Proposition (5.14). If the first ratio in (5.12) is larger than unity (the second smaller than unity), total satisfaction must have increased (decreased).

Indeed, (5.12) are nothing but the ratios

$$\frac{R_1(q_1)}{R_1(q_0)} \text{ and } \frac{R_0(q_1)}{R_0(q_0)}.$$

If the first is larger than unity, $I(q_1) > I(q_0)$ by (5.4). If the second is smaller than unity, $I(q_1) < I(q_0)$.

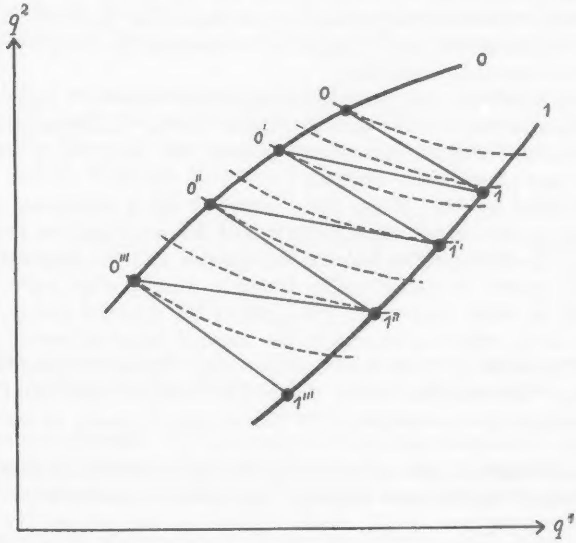

FIGURE 4

The nature of Pigou's conclusions can also be exhibited by putting them in the form:

$$\text{(5.15)} \qquad \text{if } P_{01}^{\text{La}} \leqq 1,\ q_0 \text{ 0–adapted, then } P_{01}^{\text{Ind}}(I_0) \leqq 1,$$

$$\text{(5.16)} \qquad \text{if } P_{01}^{\text{Pa}} \geqq 1,\ q_1 \text{ 1–adapted, then } P_{01}^{\text{Ind}}(I_1) \geqq 1,$$

where for brevity $I_0 = I(q_0)$, $I_1 = I(q_1)$. (5.15)–(5.16) follows immediately from (5.9) in connection with the monotonicity assumption.

Pigou's problem: measuring satisfaction variation under constant money expenditure is fundamentally different from that of constructing a functional price index, which is the problem of measuring money expenditure variation under constant satisfaction. The former problem is essentially affected by the arbitrariness of the choice-indicator (compare (4.2)), *while the latter is not.* This entails a fundamental difference in the nature of the conclusions. Indeed, if $f(x)$ and $g(x)$ are *any* monotonically increasing functions satisfying $f(1) = g(1) = 1$, $f(P_{01}^{\text{La}})$ and $g(P_{01}^{\text{Pa}})$ can be inserted for P_{01}^{La} and P_{01}^{Pa} without causing any change in Pigou's reasoning. His long analysis contains nothing to distinguish P_{01}^{La} and P_{01}^{Pa} from f and g. His use of the word "limits" is, therefore, entirely unjustified. A comparison of (5.15)–(5.16) with (5.25)–(5.26) shows how much more is contained in the propositions later developed by Haberler.

Several authors, for instance, Keynes[47] and Staehle,[48] believe that Pigou's analysis actually furnishes limits for the indifference-defined price index. These misunderstandings show how easy it is to read into other men's work one's own ideas.

The Gini identity. While Pigou searched for a barometer of the change in total satisfaction, Gini looked for one based on marginal utility. He distinguishes between the psychic and the economic purchasing power of money.[49] The former is simply some price index; usually he takes Laspeyre's. The latter is the inverted money utility ratio, ω_0/ω_1, where ω_t designates the nominal marginal money utility in the situation t, i.e., the common ratio (4.8). He also considers a third concept: the *marginal utility index* ('le nombre indice de l'utilité économique des marchandises'.)'[50] This is defined exactly as one of the

[47] In *A Treatise on Money*, Vol. 1, page 111, Keynes develops a double limit for the functional price index and says: "It is reached, for example, by Professor Pigou."

[48] In *Intern. Comp. of Food Costs*, page 75, Staehle refers to Pigou's use of the word limits and says that Pigou's theory contains the essential elements of Konüs' (1924). This cannot be said, because Konüs actually considered limits for the indifference-defined price index.

[49] *Metron*, 1924, page 17.

[50] *Metron*, 1924, page 141.

usual price indices, only with the individual marginal utilities, $u_0^1 \cdots u_0^N$ and $u_1^1 \cdots u_1^N$, inserted instead of the prices. For instance, the marginal utility indices of the Laspeyre and Sauerbeck types,

$$U_{01}^{\mathrm{La}} = \frac{\sum u_1 q_0}{\sum u_0 q_0}, \quad U_{01}^{\mathrm{Sau}} = \frac{1}{N} \sum \frac{u_1}{u_0}.$$

By virtue of the equilibrium equations under path constant prices, we get

$$\frac{\omega_0}{\omega_1} = P_{01}^{\mathrm{La}} \cdot U_{01}^{\mathrm{La}} = P_{01}^{\mathrm{Sau}} \cdot U_{01}^{\mathrm{Sau}}, \tag{5.17}$$

and similarly for any index number that satisfies the proportionality test.

Gini considers the measurement of ω_0/ω_1 as one of the main objects of index number construction.[51] From (5.17) it is seen that this ratio will simply be measured by the price index *if the average level of marginal utilities is the same in the two situations.*[52] More precisely, the price index to use in measuring ω_0/ω_1 is the one whose analogue expresses the constancy of the marginal utility level. This we may call the Gini identity.

Gini also presents another argument,[53] in my view very unsatisfactory. He uses also here the three notions defined above. The terms are exactly the same. But he cannot have the same logical content in mind. Indeed, if he had, his proposition would be:

$$\begin{gathered} \text{if } U_{01}^{\mathrm{La}} = 1 \text{ and } P_{01}^{\mathrm{La}} = \frac{\pi_0}{\omega_1}, \\ \text{then } P_{01}^{\mathrm{La}} < \frac{\omega_0}{\omega_1} < P_{01}^{\mathrm{Pa}}, \end{gathered} \tag{5.18}$$

and this does not make sense. The second equality in the premise follows from the first by (5.17) and, if it holds, the conclusion (5.18) is false.

The most plausible way to correct this argument seems to be now to interpret the "indice de l'utilité économique des marchandises" as "total satisfaction," I, and the "pouvoir économique d'achat" as (4.16). If we do, and reverse the signs, Gini's conclusion becomes tantamount to the limits developed three years later by Haberler, or

[51] *Metron*, 1924, page 22.
[52] *Loc. cit.*, page 141.
[53] *Loc. cit.*, page 148.

more precisely, to the double limit to which the Haberler limits reduce in the case of expenditure proportionality.

The Konüs limits. The first who really considered limits for the indifference-defined price index was Konüs.[54] He states explicitly the propositions (5.2)–(5.4) and uses them to prove two propositions, which we formulate thus:

$$\text{(5.19)} \quad \left.\begin{matrix}\text{If } R_1(q_1) = R_1(q_0)\\ q_1 \text{ 1-adapted}\end{matrix}\right\}\text{then } \overset{\text{La}}{P_{01}} \geqq \overset{\text{Ind}}{P_{01}}(I_1) \quad \text{(Upper Konüs limit)}$$

$$\text{(5.20)} \quad \left.\begin{matrix}\text{If } R_0(q_0) = R_0(q_1)\\ q_0 \text{ 0-adapted}\end{matrix}\right\}\text{then } \overset{\text{Ind}}{P_{01}}(I_0) \geqq \overset{\text{Pa}}{P_{01}} \quad \text{(Lower Konüs limit)}$$

where, for brevity, $I_1 = I(q_1)$, $I_0 = I(q_0)$.

Konüs proves his limits for path constant prices, but the proposition is true in general if we only adopt (5.10) and (5.11). To prove (5.19), let $\bar{q}_0$ be the point on the 0-expansion path that is indifferent to q_1, i.e., $I(\bar{q}_0) = I(q_1)$. Then,

$$\text{(5.21)} \qquad \overset{\text{Ind}}{P_{01}}(0_1) = \frac{R_1(q_1)}{R_0(\bar{q}_0)} = \overset{\text{La}}{P_{01}} \cdot \frac{R_0(q_0)}{R_0(\bar{q}_0)} \cdot \frac{R_1(q_1)}{R_1(q_0)}.$$

By the premise and (5.2), $I(q_1) \geqq I(q_0)$, hence $I(\bar{q}_0) \geqq I(q_0)$, therefore, by (4.14), $R_0(\bar{q}_0) \geqq R(q_0)$. Thus, the first fraction to the right in (5.21) is not larger than unity, and the last fraction is unity, which gives (5.19).

To prove (5.20), we only have to interchange 0 and 1 and notice that

$$\text{(5.22)} \quad \overset{\text{La}}{P_{01}}\overset{\text{Pa}}{P_{10}} = 1, \qquad \overset{\text{Ind}}{P_{01}}(I) \cdot \overset{\text{Ind}}{P_{10}}(I) = 1 \text{ (for any } I).$$

If we have expenditure proportionality, Konüs' two limits reduce to one double limit for the same number, $\overset{\text{Ind}}{P_{01}}$. But, as Konüs remarks, this cannot, in general, be assumed.

To exhibit the nature of the approximation obtained by the Konüs limits, we may write them:

$$\text{(5.23)} \quad \frac{R_1(q_1)}{R_0(q_0)} \geqq \overset{\text{Ind}}{P_{01}}(I_1) \qquad \text{if } R_1(q_1) = R_1(q_0),\ q_1 \text{ 1-adapted},$$

$$\text{(5.24)} \quad \overset{\text{Ind}}{P_{01}}(I_0) \geqq \frac{R_1(q_1)}{R_0(q_0)} \qquad \text{if } R_0(q_0) = R_0(q_1),\ q_0 \text{ 0-adapted}.$$

[54] Economic Bulletin. Conjuncture-Institute of Moscow n. 9/10, 1924 (Russian). I know his work through Bortkiewicz's account in *Nordisk Statistisk Tidsskrift*, Bd. 11, pages 18–20. Konüs has an additional argument about the difference between the signs > and ≧. Since ≧ is the only sign of interest for practical applications and, furthermore, simpler to reason with, I use this exclusively.

In other words, the upper and lower limit is the same number, namely, that which would have given the correct indifference-defined index if the two points observed *had been equivalent.* Only if this is nearly fulfilled will the approximation be a good one (compare the condition for the double limit later developed by Keynes).

The upper Konüs condition requires that the base point shall lie in the budget surface through the object point, while the lower Konüs condition requires that the object point shall lie in the budget surface through the base point. If both conditions are fulfilled simultaneously, we may say that the two points are *mutually budgetary.* They must then also be equivalent, i.e., $I(q_0)=I(q_1)$, because, by the above reasoning, $I(q_1)\geqq I(q_0)$ and $I(q_0)\geqq I(q_1)$. Furthermore, in the 0-budget surface through q_0 no point outside q_0 is equivalent to q_0 if the adaptation is unique, i.e., we must then even have $q_1=q_0$. In practice, the simultaneous fulfilment of both Konüs conditions is, therefore, a trivial case, when the points compared lie in the same indifference map.

The Haberler limits. Haberler has developed two limits that hold for any two equilibrium points without further conditions.[55] We formulate the proposition thus:

$$\text{(5.25)} \qquad \text{if } q_1 \text{ is 1–adapted, then } \overset{\text{Ind}}{P_{01}}(I_1) \geqq \overset{\text{Pa}}{P_{01}} \qquad \text{(Lower Haberler limit)},$$

$$\text{(5.26)} \qquad \text{if } q_0 \text{ is 0–adapted, then } \overset{\text{La}}{P_{01}} \geqq \overset{\text{Ind}}{P_{01}}(I_0) \qquad \text{(Upper Haberler limit)}.$$

As before, consider $\bar{q}_0$. If $\bar{q}_0$ is 0-adapted and $I(q_1)=I(\bar{q}_0)$, we must by (5.7) have $R_0(q_1)\geqq R_0(\bar{q}_0)$, which is (5.25). (5.26) is proved by interchanging 0 and 1, using (5.22). These proofs do not assume path constant prices.

If we have expenditure proportionality, Haberler's two limits reduce to a double limit. In his original analysis Haberler took this for granted, which was criticized by Bortkiewicz.[56] Haberler admits this[57] in point of principle, but thinks expenditure proportionality holds approximately for small displacements in the same indifference map.

Haberler's proposition gives an upper limit just in those cases where

[55] *Der Sinn der Indexzahlen,* 1927. The nature of the Konüs and Haberler limits are thus very different. I cannot, therefore, agree with Staehle when he says (*Intern. Comp. of Food Costs,* page 77): "Haberler's . . . theory had already been developed by . . . Konüs."

[56] *Magazin der Wirtschaft,* Berlin, 1928, page 427.

[57] *Weltwirtschaftliches Archiv,* 1929, II, page 6.

Konüs' proposition gives a lower limit, and *vice versa*. Curiously enough, this was not noticed by Bortkiewicz. Staehle first utilized it (see below).

The Keynes' double limit. Keynes[58] proved a proposition which we formulate thus:

If q_0 is 0-adapted, q_1 1-adapted, $I(q_0)=I(q_1)$, then

$$(5.27) \qquad \overset{\text{La}}{P_{01}} \geqq \overset{\text{Ind}}{P_{01}}(I) \geqq \overset{\text{Pa}}{P_{01}} \text{ where } I = I(q_0) = I(q_1).$$

This follows immediately from (5.25) and (5.26), since now $I_0=I_1$. Keynes' proof is more complicated, but correct. It implies monotonicity along a *beam* (line where all individual quantities change proportionally) instead of (4.14). In the substitution region (where the preference direction has all positive components) beam-monotonicity holds.

Later Bortkiewicz[59] and Allen [60] proved (5.27). None of these three authors noticed the perfectly trivial character of (5.27). If we know that q_0 and q_1 are adapted *and equivalent*, the indifference-defined index can be computed exactly, namely, as the ratio $R_1(q_1)/R_0(q_0)$. In these circumstances, to derive *limits* for it is to play hide-and-seek. It was Staehle who first pointed this out.[61]

The iso-expenditure method of Staehle. Utilizing at the same time Haberler's and Konüs' limits, Staehle[62] develops a method which—in a simplified and somewhat more systematized form—can be described thus: Suppose that there are given two price situations, 0 and 1, and the corresponding expansion paths (the complete Engel data for 0 and 1), Figure 4 represents the situation for $N=2$ and path constant prices. Let us start in some point q_0 on the 0-expansion path (in the figure for brevity marked 0, instead of q_0). Let us construct the 0-budget surface through this point. It intersects the 1-expansion path in a point q_1 (in the figure for brevity denoted 1). Through this point we draw the 1-budget surface. It intersects the 0-path in a point q_0'. Through this we draw the 0-budget surface. It intersects the 1-path in q_1', and so on. In two dimensions we get the zig-zag line of Figure 4. The dotted lines are indifference lines.

Let $R_0, R_0' \cdots$ and $R_1, R_1', \cdots$ be the money expenditures in the various points on the two expansion paths, $I_0, I_0' \cdots$ and $I_1, I_1' \cdots$

[58] *A Treatise on Money*, I, pages 110–111.

[59] *Nordisk Statistisk Tidsskrift*, Bd. 11, page 21.

[60] *Economica*, May, 1933, page 204.

[61] *The Review of Economic Studies*, 1935. Bortkiewicz did remark that the condition, $I(q_0)=I(q_1)$, cannot as a rule be observed (*Nordisk Statistsk Tidsskrift*, Bd. 11, page 22). But he did not see the main point, namely, that, even if it could, the limit in question would be useless.

[62] *The Review of Economic Studies*, 1935.

the indifference levels through these points. Using alternately the Konüs and Haberler limits, we then get

$$
(5.28)\qquad
\begin{array}{lll}
\dfrac{R_1}{R_0'} = P_{01}^{\mathrm{Ind}}(I_1) = \dfrac{R_1}{R_0} & & R_0(q_1) = R_0(q_0), \\[2ex]
\dfrac{R_1}{R_0'} = P_{01}^{\mathrm{Ind}}(I_0') = \dfrac{R_1'}{R_0'} & & R_1(q_0') = R_1(q_1), \\[2ex]
\dfrac{R_1'}{R_0''} = P_{01}^{\mathrm{Ind}}(I_1') = \dfrac{R_1'}{R_0'} & & R_0(q_1') = R_0(q_0'), \\[2ex]
\dfrac{R_1'}{R_0''} = P_{01}^{\mathrm{Ind}}(I_0'') = \dfrac{R_1''}{R_0''} & & R_1(q_0'') = R_1(q_1'). \\[1ex]
\cdots\cdots\cdots & & \cdots\cdots\cdots
\end{array}
$$

This is the iso-expenditure method. From a remark under the exposition of Konüs' limits it follows that it gives a good approximation only when the two expansion paths are *rather close together*. The comparison between two paths will be more exact if made via an intermediate path. The closer the individual paths the better. Knowing a very close path-system is equivalent to knowing the indifference surfaces themselves. In this last case the indifference index can be computed exactly.

6. THE THEORY OF APPROXIMATIONS

We now turn to the question of how an approximation to the indifference defined index may be obtained by some method that is not essentially connected with the study of upper and lower limits.

The Bowley approximation. One such method is given by Bowley. On an important point it needs correction. The corrected argument runs as follows. Let p_0, q_0, p_1q_1, be given, and prices path constant. Let $\bar{q}_1$ be the point on the 1-expansion path that is equivalent to q_0, i.e., $I(\bar{q}_1)=I(q_0)$. The indifference defined index is then

$$
(6.1)\qquad P_{01}^{\mathrm{Ind}}(I_0) = \frac{R_1(\bar{q}_1)}{R_0(q_0)}, \qquad \text{where } I_0 = I(q_0).
$$

The problem is to determine approximately the $\bar{q}_1$ that is equivalent to q_0. Retaining second order terms of the Taylor expansion, we get:

$$
(6.2)\qquad I(q_1) - (Iq_0) = \textstyle\sum_h I_0^h(q_1^h - q_0^h) + \frac{1}{2}\sum_{hk} I_0^{hk}(q_1^h - q_0^h)(q_1^k - q_0^k),
$$

$$
(6.3)\qquad I_1^h - I_0^h = \textstyle\sum_k I_0^{hk}(q_1^k - q_0^k),
$$

where $I_t^h=I^h(q_t)$, $I_t^{hk}=I^{hk}(q_t)$. Inserting from (6.3) into (6.2) and

noticing that under path constant prices $I_t^h = \omega_t p_t^h$, where ω_t—the nominal marginal money utility—is independent of h, we get

$$(6.4) \qquad I(q_1) - I(q_0) = \tfrac{1}{2}\sum (\omega_1 p_1 + \omega_0 p_0)(q_1 - q_0);$$

similarly,

$$(6.5) \qquad \begin{aligned} I(\bar{q}_1) - I(q_1) &= \tfrac{1}{2}\sum(\bar{\omega}_1 p_1 + \omega_1 p_1)(\bar{q}_1 - q_1) \\ &= \frac{\bar{\omega}_1 + \omega_1}{2}\sum p_1(\bar{q}_1 - q_1), \end{aligned}$$

$\bar{\omega}_1$ being the nominal money utility in $\bar{q}_1$. These two equations, together with $I(\bar{q}_1) = I(q_0)$, determine $\sum p_1\bar{q}_1$. This gives the quadric approximation

$$\overset{\text{Quad}}{P_{01}}(I_0) = \frac{\sum p_1(\omega_1 q_0 + \omega_1 q_1) + \omega_0 \sum p_0(q_0 - q_1)}{(\omega_1 + \bar{\omega}_1)\sum p_0 q_0}.$$

Rearranging the terms, this may be written

$$(6.6) \qquad \overset{\text{Quad}}{P_{01}}(I_0) = \overset{\text{Bow}}{P_{01}} + \left(\overset{\text{Quad}}{P_{01}}(I_0) - \frac{\omega_0}{\bar{\omega}_1}\right)\cdot\frac{\bar{\omega}_1 \sum p_0(q_1 - q_0)}{\sum p_0(\omega_1 q_0 + \bar{\omega}_1 q_1)},$$

where

$$(6.7) \qquad \overset{\text{Bow}}{P_{01}} = \frac{\sum p_1(q_0 + \lambda q_1)}{p_0(q_0 + \lambda q_1)}, \qquad \lambda = \frac{\bar{\omega}_1}{\omega_1}.$$

The parenthesis to the right in (6.6) will be close to zero because, as we pass from q_0 to the *equivalent* position $\bar{q}_1$, "the utility of the last dollar" will change inversely as the price level. Bowley stresses this, for the expression (6.6), but there are, in fact, many other ways to arrange the terms so as to get a remainder about which something similar can be said.

The weighted average of q_0 and q_1 in (6.7) should not be replaced by the simple average as in (2.5). This will introduce a systematic bias. For instance, if the q_1 are, on the average, much larger than the corresponding q_0, they should be weighted so as to become *still more important* (because $\bar{\omega}_1/\omega_1$ is now larger than unity). By (2.5) their importance would be attenuated. Bowley arrives at the biassed formula (2.5) because he develops (6.5) only to the first approximation, while using (6.4) to the second approximation.

If the two points q_0 and q_1 are actually equivalent, (6.7) does *not*, in general, give the correct value, $\sum p_1 q_1/\sum p_0 q_0$. This is a drawback. Furthermore, (6.7) does not bring out the *point* on the expansion paths to which the index refers.

The double expenditure method. The quadratic expansion can be utilized in another way that seems more promising. Let 0 and 1 be two given expansion paths, prices path constant. Consider a point q_0 on 0 and q_1 on 1. The condition for equivalence is, by (6.4),

$$(\omega_1\rho_1 - \bar{\omega}_0\rho_0) + (\omega_0 \sum p_0q_1 - \omega_1 \sum p_1q_0) = 0, \tag{6.8}$$

$\rho_0 = \sum p_0q_0$ and $\rho_1 = \sum p_1q_1$ being the money expenditures along the two paths. In a point of equivalence we have approximately[63] $\omega_1\rho_1 = \omega_0\rho_0$. The condition for equivalence can, therefore, approximately be written

$$\sum p_1q_1 \cdot \sum p_0q_1 = \sum p_0q_0 \cdot \sum p_1q_0. \tag{6.9}$$

The left member of (6.9), namely,

$$D_{01} = \sum p_1q_1 \cdot \sum p_0q_1, \tag{6.10}$$

we shall call the *double expenditure* along 1 (with 0 as base). It is an observable number, that may be computed in any point along 0. Similarly, D_{10} along 0. The equality between D_{01} and D_{10} indicates indifference. The square root of D_{01},

$$C_{01} = \sqrt{\sum p_1q_1 \cdot \sum p_0q_1}, \tag{6.11}$$

may be called the *crossing-expenditure* along 1. It is the geometric average of the actual expenditure along the 1-path and the expenditure that would apply here with 0 prices. C illustrates the nature of the equivalence principle (6.9), but in practical computation D will probably be more convenient.

Equating the double expenditures along the two paths, as we have done in (6.9), is the same as to define equivalence by any of the two conditions,

$$\overset{\text{La}}{Q_{01}} = \overset{\text{La}}{Q_{10}}, \qquad \overset{\text{Pa}}{Q_{01}} = \overset{\text{Pa}}{Q_{10}}, \tag{6.12}$$

the Q's being the Laspeyre and Paasche *quantity* indices. If we attempt to define equivalence by putting a quantity index equal to 1, we run into difficulties, because the usual quantity indices do not meet the point reversal test and also because Laspeyre's and Paasche's indices do not lead to the same result.[64] (6.14) is an interesting way to avoid these difficulties.

The nature of the double expenditure method is also illustrated by the fact that, in a special case, it reduces to the "ideal" formula (2.4). Indeed, let q_0 and q_1 be two fixed points (not moving along the paths). Suppose that the paths are *beams* (straight lines through origin). The

[63] Under expenditure proportionality this equation holds exactly by (7.13).
[64] See, for instance, Staehle, *Intern. Comp. of Food Costs*, page 5.

point coordinates on 0 and 1 are then $\lambda_0 q_0^h$ and $\lambda_1 q_1^h$, the variable parameters, λ, being independent of h. The λ values that equalize the double expenditures are

$$(6.13) \qquad \lambda_1/\lambda_0 = \sqrt{\sum p_0q_0 \cdot \sum p_1q_0 / \sum p_1q_1 \cdot \sum p_0q_1},$$

and the corresponding ratio between the local expenditures is $\lambda_1 \sum p_1q_1 / \lambda_0 \sum p_0q_0$, i.e., (2.4). Incidentally, this is a way to deduce (2.4). It shows that the "ideal" formula is related to the beam shape of the expansion paths.

In practice, the Engel curves are only known in discrete points. Interpolation must, therefore, be used to determine the equivalence points. Furthermore, data will be available only for a limited number of representative goods. These are the same practical difficulties as are encountered in all index number work.

The dissimilarity method of Staehle. Let q be an arbitrary and q_0 a 0-adapted complex. If the individual quantities in q are almost proportional to those in q_0, Staehle[65] says that the complexes are "similar." In this case all the deviations,

$$(6.14) \qquad \frac{q}{q_0} - \frac{\sum p_0q_0 \cdot \left(\frac{q}{q_0}\right)}{\sum p_0q_0},$$

are small. The last term in (6.14) is the weighted average of the ratios, q/q_0. Measured relative to the average itself, the deviation is

$$(6.15) \qquad \frac{q}{q_0} \cdot \frac{\sum p_0q_0}{\sum p_0q} - 1.$$

The average of the absolute values of these deviations, again using the 0 values as weights, is Staehle's measure of dissimilarity (with 0 as base),

$$(6.16) \qquad V = \frac{\sum p_0q_0 \cdot \left| \frac{q}{q_0} \cdot \frac{\sum p_0q_0}{\sum p_0q} - 1 \right|}{\sum p_0q_0} = \sum \left| \frac{p_0q}{\sum p_0q} - \frac{p_0q_0}{\sum p_0q_0} \right|.$$

Obviously, $D \geqq 0$. Further, $D \leqq 2$ because, if x and y are any two variables, $\sum|x-y| \leqq \sum|x| + \sum|y|$. Hence, $D \leqq 1+1$. Since the p's and q's are non-negative, $D=2$ when, and only when, none of the q goods occur in q_0.

Let q_0 be a fixed point on the 0-expansion path, and q another point

[65] *Archiv für Sozialwissenschaft*, June, 1932, and ECONOMETRICA, 1934, page 64.

on this path. If we start with $q=q_0$, obviously $D=0$. As q moves outwards or inwards, Staehle finds empirically that D increases practically monotonically. Thus, along the 0-path the *minimum* of D indicates that point which is "equivalent with q_0." Also, when q varies along *another path*, say 1 (while q_0 is still fixed on 0), Staehle finds empirically that D has a more or less definite minimum. The only difference is that now the minimum will not be 0 but, in general, a positive quantity. Incidentally, this may be interpreted as an irreducible dissimilarity created by the difference between the 0 and 1 price situation. In any case, the point $\bar{q}_1$ where the minimum occurs he takes as equivalent with q_0. The price index on this indifference level will, consequently, be $\sum p_1\bar{q}_1/\sum p_0q_0$. This may be done for any point q_0 on the 0-path, thus establishing a one-to-one correspondence between points on the 0 and 1 paths.

Graphically, D may be represented as a surface over the $(\rho_0\rho_1)$ plane. The "valley of dissimilarity" marks the equivalent combination of ρ_0 and ρ_1. In practice, the valley will, of course, not be defined with mathematical exactness, but still with sufficient accuracy to give an idea of the incomes that are equivalent.[66]

7. THE FLEXIBILITY METHOD[67]

Let I and J be two indicators, i.e., $I=F(J)$, F being monotonically increasing. Then dI/dJ, $(d \log I)/(d \log J)$, $(d \log (dI/dJ))/(d \log J)$, etc., are indifference functions. They are even indicators if they change monotonically with I (or J). It may be difficult to observe I or J directly as functions of q over the entire space, but sometimes one of the derived functions may be accessible to measurement, thus giving an observable criterion for equivalence of expenditure even between remote points. This is the basis of the flexibility method I suggested three years ago.[68] The method may be formulated in more general terms than I used originally, thus meeting certain objections.[69]

The definition of real expenditure. Let $r_0(I)$ be a function—for the moment chosen conventionally—expressing real expenditure along the 0-path taken as base, for instance,

$$(7.1) \qquad r_0(0) = \rho_0(I) = \text{money expenditure along base–path}.$$

Real expenditure, r, in any point, q, may then be defined as

[66] See ECONOMETRICA, 1934, pages 67 and 68.

[67] My thanks are due to Mr. Tjalling Koopmans of Amsterdam, *pro. tem.* Oslo, who has kindly read Section 7 in MSS. and suggested several improvements in the presentation.

[68] *New Methods of Measuring Marginal Utility*, Tübingen, 1932. Section 9.

[69] In particular those made by Allen, *Economica*, May, 1933.

$$r = r_0(I), \qquad \text{where } I = I(q). \tag{7.2}$$

This r will be an indifference function, even an indicator if $r_0(I)$ is monotonically increasing. Let $\bar{r}$ be the concept obtained by using 1 as base, $r_1(I)$ the conventional function along 1. Then

$$\frac{\bar{r}}{r} = \frac{r_1(I)}{r_0(I)}. \tag{7.3}$$

If it is possible to formulate the convention in such a way that (7.3) becomes independent of I, and further r, respectively $\bar{r}$, a plausible expression for real expenditure, *that* particular convention ought to be adopted. In the case of income proportionality this leads to (7.1). Otherwise, (7.1) is more or less arbitrary. Whatever the convention, r is an indifference function.

The problem may also be approached as one of *deflation:* Can we find a number which, divided into any ρ_t, gives the corresponding r? In the expenditure proportionality case, P_{0t}^{Ind} is such a number, but not in the general case. Indeed, using P_{0t}^{Ind} here would mean that on the conventionally chosen 0-path by postulate the "rich man's price level" is equal to the "poor man's price level," while on any other path it need not be (P_{0t}^{Ind} is always independent of I, but P_{0t}^{Ind} is not). To treat the general case adequately, we must introduce also a deflation factor, $P_0(I)$, for comparisons along the base path itself. On 0 we then have

$$r_0(I) = \frac{\rho_0(I)}{P_0(I)}. \tag{7.4}$$

Thus, $r_0(I)$ and $P_0(I)$ are only two ways of expressing the real expenditure convention. Most of the time we shall use $P_0(I)$. (7.1) means putting $P_0(I) \equiv 1$.

The deflation factor on any other path, t, will be

$$P_t(I) = P_0(I) \cdot P_{0t}^{\text{Ind}}(I). \tag{7.5}$$

Deflating ρ_t by P_t, we get the *same* r as by (7.2). This ratio—the real expenditure—is, therefore, an indifference function.

$$r(I) = \frac{\rho_0(I)}{P_0(I)} = \frac{\rho_1(I)}{P_1(I)} = \cdots \frac{\rho_t(I)}{P_t(I)} = \cdots. \tag{7.6}$$

Since I is a function of r, ρ_t and P_t are also. Using this, we get by logarithmic differentiation of (7.6),

$$\frac{d\log \rho_t}{d\log r} = 1 + \frac{d\log P_t}{d\log r} = \frac{1}{1 - \dfrac{d\log P_t}{d\log \rho_t}}. \tag{7.7}$$

The derivative,

$$\frac{d\rho_t}{dr} = P_t\left(1 + \frac{d\log P_t}{d\log r}\right), \tag{7.8}$$

may be called the marginal *path expendivity*. It is analogous to (4.10) and expresses the money cost of acquiring an additional unit of real expenditure.

Putting, for brevity,

$$P_t = \frac{d\log P_t}{d\log \rho_t}, \tag{7.9}$$

we have

$$\frac{d\log \rho_1}{d\log \rho_0} = \frac{1 - \breve{P}_0}{1 - \breve{P}_1} \qquad (\text{if } \rho_1 \text{ equiv. } \rho_0). \tag{7.10}$$

This follows from (7.7), because $P_{0t}^{\text{Ind}} = \rho_1/\rho_0$ (if ρ_1 equiv. ρ_0).

Money utility and money flexibility. Let q be any point and dq a small displacement in the direction $(dq^1 \cdots dq^N)$. In general, the ratio $dI/d\rho_t$ will depend on the direction. But, if q is a t-adapted point, it will not. It is then simply the *nominal* money utility, ω_t (the common ratio (4.8)), i.e.,

$$\omega_t = \frac{dI}{d\rho_t} \quad (d \text{ taken in any direction from a } t\text{–adapted point}). \tag{7.11}$$

Indeed $d\rho_t = \sum_h (\partial\rho_t/\partial q^h)dq^h$, which, by (4.8), is $\omega_t \sum I^h dq^h = \omega_t dI$.

In analogy with (7.11), consider the *real* money utility,

$$w = \frac{dI}{dr} = w(r). \tag{7.12}$$

It is an indifference function. By (7.8),

$$\omega_t = \frac{w(r)}{\dfrac{d\rho_t}{dr}} = \frac{w(r)}{P_t\left(1 + \dfrac{d\log P_t}{d\log r}\right)}. \tag{7.13}$$

This shows that, besides the weighted "marginal utilities" of the individual goods (4.8), we may consider another—in the equilibrium equal to the rest—namely, the weighted utility of real expenditure.

It is completely analogous to that of the individual goods. Compare the denominator to the extreme right in (7.13) with (4.10).[70] The analogy, however, only appears when we consider the general case of not-path-constant individual prices. In practice, the individual prices are, as a rule, path constant, but P_t not. This may mislead us by false analogy to define real money utility, w, by

$$(7.14)\quad \omega_t = \frac{w(r)}{P_t}, \quad \text{which reduces to} \quad \omega_t = \frac{w(r)}{P_{0t}} \quad \text{when } P_0 \equiv 1.$$

If we do, w will in general *not* be an indifference function. The statistical technique will then be cumbersome and inelegant (unless we have expenditure proportionality).

Furthermore, we shall consider the money flexibility,

$$(7.15)\qquad \breve{w} = (\breve{w}r) = \frac{d\log w(r)}{d\log r}.$$

It is an indifference function, since w and r are.[71] Empirically, it turns out to be changing monotonically with r, so that it is even an indicator. Hence, equivalence of expenditure can be measured by the equality of the money flexibility.

$\breve{w}$ may be expressed in terms of t data; we only have to express $d\rho_t/dr$ in (7.13) by $\breve{P}_t$ and take the logarithmic derivative. This gives

$$(7.16)\qquad \breve{w} = \frac{\breve{\omega}_t \quad \breve{P}_t}{1-\breve{P}_t} \quad \frac{\dfrac{d\breve{P}_t}{d\log \rho_t}}{(1-\breve{P}_t)^2} \qquad (d \text{ taken along the } t\text{-path}),$$

where

$$(7.17)\qquad \breve{\omega}_t = \frac{d\log \omega_t}{d\log \rho_t} \qquad (d \text{ taken along the } t\text{-path}).$$

(7.16) shows that the money flexibility, $\breve{w}$, taken with regard to real expenditure, is, in general, not the same as the nominal flexibility, $\breve{\omega}_t$, taken along the t path. Even if individual prices are path constant, P_t

[70] (7.13)—here derived as a theoretical consequence—should completely meet Allen's objection, *loc. cit.*, middle of page 193. (7.13) shows that my original formula does hold under expenditure proportionality, which was assumed in the statistical work in *New Methods* . . . .

[71] I cannot agree with Allen, who characterizes this as an "impossible condition" (*Economica*, May 1933, page 208). On the contrary, I find it very plausible that w should be an indifference function. The above analysis even shows that this is not a condition but a theoretical consequence that follows quite generally when the appropriate system of definitions is adopted.

may not be and, consequently, by (7.16), $\breve{w}$ and $\breve{\omega}_t$ different.[72] $\breve{\omega}_t$ is independent of the conventional function P_0, but $\breve{w}$ is not.

The independent reference set. To develop a method of actually measuring $\breve{w}$, let us divide the complete set of N goods in two subsets, for brevity "non-food" (goods Nos. 1, 2, $\cdots$, n) and "food" (Nos. $n+1, \cdots, N$) The quantities of the latter we denote $x^1, x^2, \cdots, x^m$ $(m=N-n)$.

Suppose that the indicator can be transformed to

$$(7.18) \qquad I(q^1 \cdots q^N) = V(q^1 \cdots q^n) + U(x^1 \cdots x^m),$$

V and U depending only on the variables indicated. I then say that $x^1 \cdots x^m$ is an independent reference set. If such a transformation exists, it is uniquely determined apart from an arbitrary *linear* (monotonically increasing) transformation.[73] Indeed, if $F(V+U)=\bar{V}+\bar{U}$, $\bar{V}$ being independent of $x^1 \cdots x^m$ and $\bar{U}$ of $q^1 \cdots q^n$, then $F'(U+V)=(\partial\bar{U}/\partial x^h)/(\partial\bar{U}/x^h)$ must be independent of $q^1 \cdots q^n$, which—when V actually depends on $q^1 \cdots q^n$—entails $F'=$constant. The argument applies even if any of the subsets consists of a single commodity only.

Suppose that the supply-price functions for the food set are independent. Let $H_t^h(x^1 \cdots x^m)$ be the price of good No. h that prevails in t when $x^1 \cdots x^m$ are taken. The adaptation within the food set is then—when total food expenditure ξ_t is given—completely determined, i.e., independent of the other data. We may, consequently, define food quantity, x, marginal food utility, $u(x)$, and total food price, H_t, just as we have defined r, $w(r)$, and P_t. H_t will be a function of x. Similarly, P_t is a function of r. This leads to[74]

$$(7.19) \qquad \frac{w(r)}{P_t(r)\left(1+\dfrac{d\log P_t(r)}{d\log r}\right)} = \frac{u(x)}{H_t(x)\left(1+\dfrac{d\log H_t(x)}{d\log x}\right)},$$

which is analogous to the equilibrium equation for two individual goods. By (7.19), x becomes—under given t—a function $x=E_t(r)$. Therefore,

$$(7.20) \qquad w(r) = \alpha_t(r)\cdot u(x),$$

where α_t is a function of r,

[72] Allen, *loc. cit.*, p. 207, seems to argue as if w and ω_t were the same: "A second derivative . . . prices being constant . . . ."

[73] This question as well as others regarding the topography of the choice field is very unsatisfactorily treated by Pareto.

[74] $P_t(\quad)$ denotes in (7.19) dependence on r and in (7.5) on I. This need not cause confusion.

$$(7.21)\qquad \alpha_t(r) = \frac{P_t(r)\left(1 + \dfrac{d\log P_t(r)}{d\log r}\right)}{H_t(x)\left(1 + \dfrac{d\log H_t(x)}{d\log x}\right)} = \frac{P_t(r)}{1 - \breve{P}_t(r)} \cdot \frac{1 - \breve{H}_t(x)}{H_t(x)},$$

$$x = E_t(r),$$

$\breve{H}_t$ being—in analogy with $\breve{P}_t$—the logarithmic derivative of H_t with respect to ξ_t. If the relative price indices, P_{0t} and H_{0t}, are determined by some approximation method, for instance Staehle's iso-expenditure method or my double-expenditure method, and one of the P functions, for instance $P_0(t)$, and one of the H functions, for instance $H_0(t)$, are chosen conventionally, r and x are known, $E_t(r)$ is, therefore, an observable Engel function and, consequently, $\alpha_t(r)$ an observable function for each t.

For real incomes, r_0 and r_1 in 0 and 1, such that $x_0 = x_1$, we get, by (7.20),

$$(7.22)\qquad \frac{\log w(r_0) - \log w(r_1)}{\log r_0 - \log r_1} = \frac{\log \alpha_0(r_0) - \log \alpha_1(r_1)}{\log r_0 - \log r_1} \qquad (\text{if } x_0 = x_1).$$

Formula (7.22) gives a measurement of the average flexibility w over the $(r_0 r_1)$ range. If r_0 and r_1 are sufficiently close, we have approximately a point measurement of $\breve{w}$. This is a generalization of the isoquant method[75] to the case where expenditure proportionality is not assumed.[76]

The right member of (7.22) is independent of the conventional function $H_0(x)$. Indeed, the H terms in the numerator of (7.22) are

$$(7.23)\qquad \log\frac{1 - \breve{H}_0}{H_0} - \log\frac{1 - \breve{H}_1}{H_1} = \log\frac{1 - \breve{H}_0}{1 - \breve{H}_1} + \log H_{01}.$$

By (7.10)—the analogue of which holds for H because ξ_1 equiv. ξ_0—(7.23) reduces to $\log(\xi_1/\xi_0 + d\log \xi_1/d\log \xi_0)$, which is independent of any arbitrary H function. A similar reduction does not take place with the P terms.

Distant Comparisons. The flexibility, w, thus measured can serve to make distant price-comparisons. Consider four paths, 0 and 1 close together, 2 and 3 also close together but far away from the first pair. A relative price index P_{01} may be constructed by one of the above approximation methods. It is a function, say, of $\rho_0 P_{01}(\rho_0)$. Similarly,

[75] *New Methods*, page 35.

[76] A paper giving a more detailed exposition of the generalized method with graphs, etc., will soon appear elsewhere.

$H_{01}(\xi_0)$, $P_{23}(\rho_2)$, and $H_{23}(\xi_2)$. But the approximation methods do not give any index P_{02}.

Consider the path flexibilities along 0 and 2. They are connected by the relation,

$$(7.24)\qquad 1 + \breve{\omega}_0 = \frac{1 + \breve{\omega}_2}{1 - \breve{P}_{02}} \quad \frac{\dfrac{d\log \breve{P}_{02}}{d\log \rho_2}}{(1 - \breve{P}_{02})^2} \qquad \text{(for equivalent points on 0 and 2)}.$$

Indeed, let $\breve{w}_0$ be the real money flexibility obtained by putting $P_0 = \text{constant}$. It is an indifference function. Along the 0 path it is, by (7.16), equal to $\breve{\omega}_0$. Along 2 its value is obtained by inserting into (7.16) $P_2 = P_0 P_{02}$, where $P_0 = \text{constant}$. Equating the two expressions for $\breve{w}_0$, we get (7.24).

In (7.24), both $\breve{w}_0$ and $\breve{\omega}_2$ are observable. Indeed, $\breve{\omega}_0$ is equal to $\breve{\omega}_0$ which—since P_{01} and H_{01} are known—can be computed by (7.22), using the data along 0 and 1. The $\breve{w}_0$ values thus obtained are by (7.23) independent of the arbitrary H_0 function. (The arbitrary P_0 function is put constant.) Similarly, $\breve{\omega}_2$ is equal to the flexibility measure $\breve{w}_2$ obtained by putting $P_0 = \text{constant}$ and using (7.22) on the data along 2 and 3. (This P_2 is, in general, different from the one corresponding to $P_0 = \text{constant}$.) The $\breve{w}_2$ values thus obtained are independent of the arbitrary H_2 function.

P_{02} in (7.24) is not known but will be so as soon as equivalence between points on 0 and 2 is determined. And this, in turn, is defined by the fact that the equation is fulfilled. Consequently, it should be possible to determine equivalence by an iteration process.

If there is expenditure proportionality between 0 and 2, (7.24) gives

$$(7.25)\qquad \breve{\omega}_0 = \breve{\omega}_2.$$

The point correspondence between 0 and 2 defined by (7.25) seems, therefore, a plausible first approximation. Let $P_{02}^{(1)}$ be the index to which it leads. Inserting this in the right member of (7.24), we get

$$(7.26)\qquad \Omega_{02}^{(1)} = \frac{1 + \breve{\omega}_2}{1 - \breve{P}_{02}^{(1)}} + \frac{\dfrac{d\log \breve{P}_{02}^{(1)}}{d\log \rho_2}}{(1 - \breve{P}_{02}^{(1)})^2},$$

which may be computed in any point along 2. Next consider the point correspondence defined by

$$(7.27)\qquad 1 + \breve{\omega}_0 = \Omega_{02}^{(1)}.$$

It leads to an index $P_{02}^{(2)}$ which, inserted in the right member of (7.24), gives a function $\Omega_{02}^{(2)}$ that may again be compared with $1+\breve{\omega}_0$, etc. If the process converges, we get the indifference index between 0 and 2.

Since the only thing compared between 0 and 2 is the magnitude of a flexibility, which is a pure number, the method may formally be applied even if the two pairs do not lie in the same map. They may represent entirely different populations, with altogether different goods, etc. But, of course, (7.24) will then only have a more or less heuristic value. The fact that it leads to the correct indifference index, if the two pairs do lie in the same map, is, however, a strong point in its favor.

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COMPOSITE COMMODITIES AND THE PROBLEM OF INDEX NUMBERS[1]

By Wassily Leontief

1. Modern economic theory is making ever increasing use of index numbers. This is a logical result of the predominant tendency toward quantitative analysis. In a loose qualitative description, such terms as "real wage" and "producer's goods" may simply indicate the totality of commodities which have certain characteristics in common. But this simple interpretation fails to satisfy the theorist when he tries to find definite functional relations, for example, the supply and demand curves of these commodities. The complicated algebraic formulae of modern monetary theory are evidently built on the assumption that composite commodities have exactly the same definitely measurable dimensions of quantity, price, utility, etc., as any of the individual commodities.

Nobody would contest the practical difficulty of measuring composite prices and quantities. This circumstance certainly reduces the practical usefulness of the abstract deductions in which they are introduced; it would, however, hardly imperil their theoretical significance as long as we assume that these composite quantities and prices exist at least as theoretical realities. The strong belief in their existence certainly constitutes one of the most important items in the credo of some present day theorists.

The important practical task of measuring these collective prices and quantities is usually left to the statistician. Turning to the statistical literature devoted to the problem of index numbers, we find, however, a development directly antithetical to the increasing optimism of the general theorist. Assigned the task of measuring certain objects, the statistician became doubtful of their very existence. What appeared to be a practical difficulty seems to reveal itself as a logical impossibility.

The theorists readily made certain concessions. Many of them abandoned the notion of a general price level. But this served only to transfer the emphasis to partial price levels. Such an attempt to meet the

[1] This article was completed more than a year ago. In the fall of 1933 it was communicated to Professor Schumpeter's Discussion Group at Harvard and on June 24, 1935 it was presented at the meeting of the Econometric Society in Colorado Springs.

The June 1935 issue of the *Review of Economic Studies* contains a paper on Index Numbers by Dr. Hans Staehle in which the treatment of several topics is very similar to that given in the present article.

difficulty half way cannot solve the problem. The theoretical problem of lumping together several commodities is essentially the same whether their number be 5000, 50, or only two.

Many misunderstandings in the discussion of index numbers seem to be the result of an insufficient discrimination between statistical and theoretical aspects of the problem. The most often repeated statement to the effect that one or another index formula gives an approximate measurement of "real wages," or "retail prices," and so on, can have two totally different meanings. It may indicate that due to inadequacy of data or imperfect statistical technique the result does not represent the exact measurement of the unknown but nevertheless theoretically existing true value of the "real wage" or "general retail price," or it may mean, in addition, that the unknown ideal value at which the statistical computation is aimed lacks in itself theoretical precision.

2. Keynes' *Treatise on Money* contains an excellent comprehensive survey of established opinion on index numbers and may be used advantageously as a point of reference in the following discussion. The larger part of his presentation is devoted to a specific composite commodity—"consumers' goods"—and he discusses predominantly the reciprocal of its composite price—the "purchasing power of money." But it is evident that the arguments of Chapter 8, Vol. I of the *Treatise* must apply to composite goods in general.[2] The theoretical conclusions of this discussion may be stated in the following points:

1. An exact computation of the composite price of consumers' goods is limited to
   a. persons of similar tastes,
   b. and among them only those who receive equal real incomes.
2. Under these conditions the composite prices of consumers' goods for different persons or for the same person at different times is proportional to the total monetary expenditures.

In order to elucidate the exact meaning of this statement and at the same time to provide a precise tool for our further analysis, we may resort to a graphic representation. The illustration can be simplified without impairing the generality of the results if we assume first that only two goods, say bread and meat, enter the consumer's budget.

The tastes of a consumer may be represented in a two-dimensional diagram by a series of indifference lines, as in Figure 1.

Measuring the quantity of meat along OB, the ordinate, and the

[2] See also H. Staehle, *International Comparison of Food Costs, A Study of Certain Problems Connected with the Making of Index Numbers.* Reprinted from Studies and Reports of The International Labor Office, Series N, No. 20.

quantity of bread along the abscissa, OA, we can represent all the possible combinations of these two commodities by corresponding points on the plane AOB. Each of the successive indifference curves (I, II, III, etc.) connects points of equal utility, i.e., all the equivalent bread and meat combinations which, in the estimate of the given person, are considered equal in their utility.

The points which are situated on different indifference lines evidently indicate combinations of unequal utility. Although it is impos-

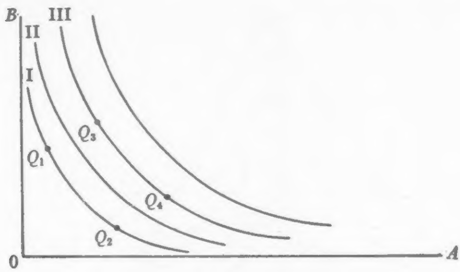

FIGURE 1

sible to measure the absolute amounts of satisfaction, the relative position of the indifference lines indicates the "more" or "less" of corresponding utilities.

Similarity of tastes, which is the first theoretical condition (1a) for the calculation of a price index, means identity in the systems of indifference lines. A composite price which is calculated for a given system of indifference lines has no significance whatever for any other differently shaped system. The second condition (1b) concerning equality of real incomes means that only such combinations of meat and bread as are situated on the same indifference lines (for example, Q_1 and Q_2, or Q_3 and Q_4, in Figure 1) can be lumped together in a single composite commodity.

Following the definition of a price index given above, we can derive the composite price relation between two equivalent combinations of A and B graphically. In Figure 2 a third dimension, M, is introduced and used for measurement of the total monetary income (=expenditure). The points Q_1 and Q_2, situated on the same indifference line $i-i$, represent the two equivalent combinations of the commodities A and B whose prices are to be compared. The slopes, bo/co and do/ao of the two tangents cd and ab indicate the relative prices, i.e., the exchange ratios between the commodities A and B in the two "positions" Q_1 and Q_2,

respectively. If the slope (fo/ao) of the line, fa, corresponds to the price of the commodity B in the position Q_2, fo (measured along the monetary coordinate M) will represent the total expenditure of the consumer in this position, i.e., the total price of the combination Q_2, or Σp_2q_2 according to the usually accepted notation.[3] The same line of reasoning shows that if the slope of the line ce, (eo/co), represents the price of B in position Q_1, eo gives the total actual expenditure for the combination Q_1 (Σp_1q_1). Finally, according to the foregoing definition of a price index, we find that the relation fo/eo or $\Sigma p_2q_2/\Sigma p_1q_1$

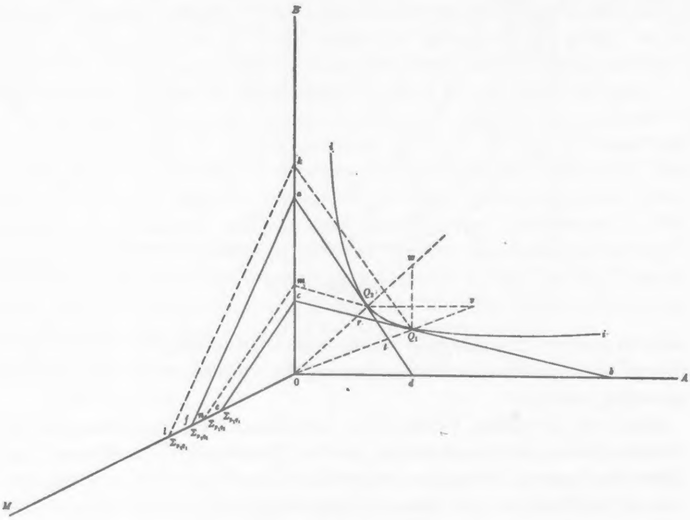

FIGURE 2

measures exactly the change of the composite price between the positions Q_2 and Q_1.

Figure 3 represents a case in which Q_1 and Q_2 are situated in the same indifference system, but on separate indifference curves, Q_1 on $i-i$ and Q_2 on $j-j$. Under this condition two theoretical price comparisons can be made; one on the level of the indifference curve $i-i$ and another on the level of the curve $j-j$. In order to find these indices, two additional points of comparison must be determined: point $\bar{Q}_1$ in-

[3] Given the price of B and the exchange value of A in B, we can derive the corresponding price of A, which is given for the position Q_1 by the slope of the line fb, fo/bo.

dicating the quantities of A and B which would be consumed on the $i-i$ utility level *under the price conditions of* Q_2, and $\bar{Q}_2$ indicating the position to which the consumer would have to shift in order to attain the utility level $j-j$ *with prices which are prevailing in* Q_1. Subsequently, the price level of $\bar{Q}_1$ can be compared with Q_1, and $\bar{Q}_2$ with Q_2. In both cases the problem is identical with the previously described comparison of two equivalent combinations of A and B. The result

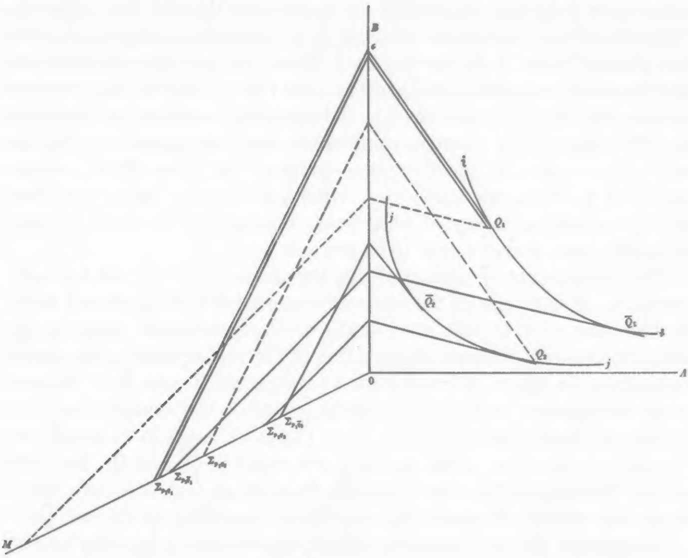

FIGURE 3

of this calculation is two different and theoretically independent price indices, $\Sigma p_1q_1/\Sigma p_2\bar{q}_1$ and $\Sigma p_1\bar{q}_2/\Sigma p_2q_2$. The independence of these two results must be stressed particularly in view of the frequent misunderstandings it has been subject to. Neither of the two ratios really gives a comparison of the price levels in Q_1 and Q_2. Such an interpretation would be possible only on the assumption that the price levels in $\bar{Q}_2$ and $\bar{Q}_1$ are equal to the price levels in Q_1 and Q_2, respectively. This would mean that the price change between $\bar{Q}_2$ and Q_1 is necessarily equal to the price difference between $\bar{Q}_1$ and Q_2, which evidently is disproved by the double result of the previous calculation.

There seems to remain, however, one possibility to circumvent this objection and to extend the comparability of composite prices beyond

the narrow limits of a single indifference curve to the full range of the system. Very often the opinion is expressed that, although an absolutely general comparison of price levels is unattainable, it is still possible to cover the whole range of changes "from the point of view" of a given income level. By means of such a partial approach some authors hope to overcome the objection which was formulated in the preceding paragraph. According to their interpretation, the comparison of composite prices in Q_1 and Q_2 "from the point of view" of the income level $j-j$ can reasonably be made *only* through the point $\bar{Q}_2$. The alternative calculation through $\bar{Q}_1$ is rejected as being made from the point of view of the income $i-i$. Thus, the paradoxical double result is readily avoided, and nothing seems to contradict the previous assumption that the price level in $\bar{Q}_2$ is identical with the price-level in Q_1. With the help of a similar approach it would be possible to find for any other—even the most remote—point of the plane BOA a corresponding point on the curve $j-j$. With the aforementioned qualifications, the comparability of price levels throughout the whole system of indifference curves seems to be proved.

The fundamental weakness of this argument, however, can be easily revealed. If the price of the composite commodity in $\bar{Q}_2$ should really be the same as in Q_1, the ratio of the total expenditures, $\Sigma p_1q_1/\Sigma p_2\bar{q}_2$, would necessarily indicate the relation of the corresponding composite quantities. In Figure 4, in addition to the points Q_1 and $\bar{Q}_2$ of the previous comparison, we introduce points $\bar{\bar{Q}}_2$ and Q_3, the second being situated on a third indifference line, $e-e$. The price levels in Q_3 and $\bar{\bar{Q}}_2$ are "equal" in the same sense as they are equal in $\bar{Q}_2$ and Q_1, the ratio of the total expenditures in Q_3 and $\bar{\bar{Q}}_2$, $\Sigma p_3q_3/\Sigma p_3\bar{\bar{q}}_2$ (equal lo/no), represents the relation between the composite quantities in Q_3 and $\bar{\bar{Q}}_2$.

Comparing the two quantity indices, we see that according to Figure 4

$$\frac{\Sigma p_1q_1}{\Sigma p_1\bar{q}_2} < \frac{\Sigma p_3q_3}{\Sigma p_3\bar{\bar{q}}_2}.$$

"From the point of view" of the income level $j-j$, the quantity of the composite commodity in $\bar{\bar{Q}}_2$ and $\bar{Q}_2$ is necessarily the same. It follows that the quantity of the composite commodity in Q_3 is larger than in Q_1. Point Q_1 is situated, however, on a "higher" indifference curve than point Q_3, which indicates that the utility in Q_3 is lower than in Q_1. This shows that the previous conclusion is wrong, because in a fixed system of taste a larger quantity can in no case represent a smaller utility. It has been proved that even "from the point of view" of a fixed income level the validity of the usual index numbers is in general

strictly limited to comparisons between points which are situated on the same utility level.

In concluding this critical survey, it has to be pointed out that only on rare occasions will the actual index calculations lead to such openly

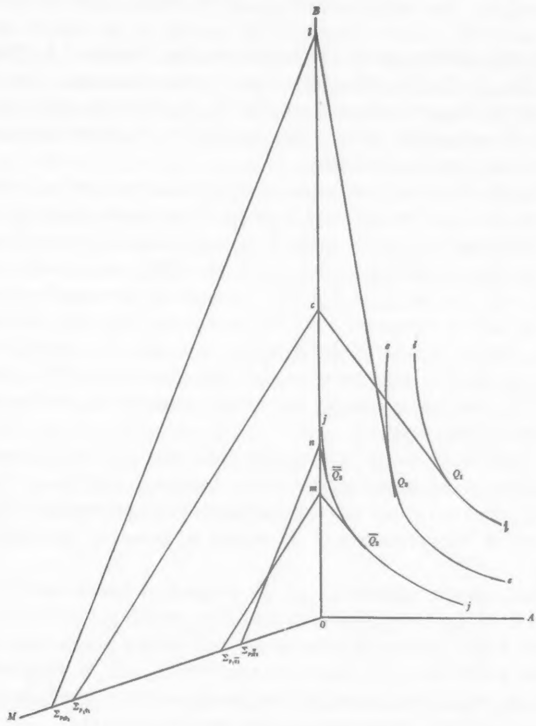

FIGURE 4

contradictory results. In a vast majority of practical cases these results are quite plausible, which does not prove, of course, that they have a clear theoretical meaning.[4] In such instances one may speak of statistical approximations to a theoretically indeterminate concept. Indeed,

[4] Under special circumstances, for example in case the indifference curves are "degenerated" in a system of parallel straight lines or at least can be approximated by such a system, these results will even have a definite theoretical meaning. But in such symmetrical cases any index calculation becomes rather superfluous.

a proper compromise between statistical optimism on one hand and theoretical purism on the other seems under these circumstances to be inevitable. But, in view of this necessity, a clear statement of the theoretical side of the case appears to be particularly important.

II

3. After discussing many different formulae, Keynes (*A Treatise on Money*, Vol. I, Ch. 8) ultimately comes to the conclusion that two of them may be theoretically and practically eligible for use in the measurement of composite prices—the method of "highest common factors" and the "method of limits."

The first method is identical in its theoretical concept with the well-known method of "equal food baskets." Its weaknesses have often been pointed out. Putting it in the language of our diagrams (Figure 1), the theoretical condition under which the "highest common factor" method could precisely be applied amounts to the requirement that the two points of comparison should be situated not only on the same curve in identical sets of indifference lines, but in addition should occupy exactly the *same point* on this identical curve. All the advantages of its practical simplicity can hardly compensate for the extreme limitations of this method.

The "method of limits" has received the most favorable consideration in theoretical literature on index numbers and its application, seemingly, does not imply any special theoretical assumptions. The usually accepted interpretation of its results is, however, somewhat misleading.

The first step in application of the method of limits consists in the calculation of the hypothetical expenditure which the person situated in Q_1 (see Figure 2) would incur if he had to buy goods according to the prices which actually existed in position Q_2, and, analogously, the expenditure which would result if the combination of goods in point Q_2 had to be paid for according to the prices of position Q_1.

In order to find the hypothetical expenditure in point Q_1, we draw Q_1k parallel to Q_2a and kl parallel to af; lo is then the total expenditure for Q_1 evaluated according to Q_2 prices. Similarly, we draw Q_2mn parallel to Q_1ce and find that no is the expenditure for Q_2 according to the prices in Q_1. Following the generally accepted algebraic notation, lo can be replaced by Σp_2q_1, fo by Σp_2q_2, no by Σp_1q_2, and eo by Σp_1q_1.

Now it is possible to show that the ratio between the hypothetical and actual price of Q_1, $\Sigma p_2q_1/\Sigma p_1q_1$, cannot be smaller than the "true" index ratio $\Sigma p_2q_2/\Sigma p_1q_1$ and, on the other hand, that the ratio $\Sigma p_2q_2/\Sigma p_1q_2$ in no case can be larger than $\Sigma p_2q_2/\Sigma p_1q_1$. In other words, under the

given assumptions, $\Sigma p_2q_1/\Sigma p_1q_1$ indicates the upper, and $\Sigma p_2q_2/\Sigma p_1q_2$ the lower limit of the "true" price change, $\Sigma p_2q_2/\Sigma p_1q_1$.

The practical significance of this result appears, however, to be rather small if we realize that under the given assumptions the available data on total monetary expenditures in the two positions of comparison enable us to derive directly and exactly the true relation, $\Sigma p_2q_2/\Sigma p_1q_1$, without the calculation of any limits. The importance of this consideration is enhanced by the fact that when these assumptions are not fulfilled, the method of limits ceases to function properly. It has been demonstrated previously that, if two income points are situated on different income levels, a theoretical computation leads to two different ratios. It is evident that the double result which is yielded by the method of limits cannot coincide with the two true theoretical ratios. But there still remains the hypothetical possibility that the statistical calculation indicates the limits between which both theoretical indices are located. The following consideration shows, however, that this is not the case:

The theoretical comparison of Q_2 and Q_1 (Figure 3) leads, as was previously mentioned, to the double result $\Sigma p_2q_2/\Sigma p_1\bar{q}_2$ and $\Sigma p_2\bar{q}_1/\Sigma p_1q_1$. Applying the method of limits to the given positions Q_2 and Q_1 in the same way as was demonstrated in Figure 2, we get (Figure 3) the supposed lower and upper "limits" $\Sigma p_2q_2/\Sigma p_1q_2$ and $\Sigma p_2q_1/\Sigma p_1q_1$.

From the relative positions of all points involved in the argument, we derive the following inequalities:

$$\frac{\Sigma p_2q_2}{\Sigma p_1\bar{q}_2} > \frac{\Sigma p_2\bar{q}_1}{\Sigma p_1q_1} \quad \text{and} \quad \frac{\Sigma p_2q_2}{\Sigma p_1q_2} < \frac{\Sigma p_2q_1}{\Sigma p_1q_1},$$

$$\frac{\Sigma p_2q_2}{\Sigma p_1q_2} < \frac{\Sigma p_2q_2}{\Sigma p_1\bar{q}_2} \quad \text{and} \quad \frac{\Sigma p_2q_1}{\Sigma p_1q_1} > \frac{\Sigma p_2\bar{q}_1}{\Sigma p_1q_1}.$$

This means that the upper "limit" of the empirical calculation is higher than the lower value of the theoretical relation and that the lower empirical "limit" is smaller than the higher theoretical. Consequently, the theoretical values may with equal probability be situated outside or within the empirical "limits." In other words, the two additional points Σp_1q_2 and Σp_2q_1 are of no help in the determination of the position of the true theoretical values.

Summarizing this preliminary analysis of the method of limits, we may say that, in cases in which it is theoretically correct, it appears to be useless practically, and in the other cases in which its application seems to be desirable, its results are faulty or, at least, inconclusive.

Unlike Keynes, Dr. Haberler (*Der Sinn der Index-Zahlen*, Tübingen

1927, pp. 94–96) has noticed the possibility of such an apparent contradiction, but in discussing this question he misses the real clue to the problem and, instead of tracing the difficulty to the incompatibility of unequal utility levels (real incomes), he holds the difference in monetary (nominal) incomes of the two positions responsible for this failure.

It is easy to see that Figure 3 can be modified in such a way that point Σp_1q_1 will be coincident with point Σp_2q_2 (which means equality of monetary income of the two positions) without affecting any of the inequalities stated above, i.e., without eliminating the fundamental difficulty.

4. What is the source of all these contradictions?

According to its fundamental set-up, the "method of limits" (and with it also all the other methods of index calculation which are mentioned in Dr. Keynes' survey) is based on the assumption that the price as well as the amount of a composite commodity is a *non-measurable magnitude.*

Such magnitude cannot be measured, i.e., described as multiple of some appropriately chosen units (dollars, pounds, yards, etc.). Given two magnitudes of this kind, it is, however, possible to tell whether they are equal or not and, in the latter case, which is the larger and which the smaller one. The ratio of two non-measurable magnitudes equals 1 if they are identical, otherwise no definite numerical value can be attached to it. In general, such quotients can be described only as being larger or smaller than one ($\gtrless 1$). A series of indifference lines is a typical example of this kind of interrelation. The order of their increasing or decreasing magnitudes (higher or lower utility levels) is well established, but there is no sense in asking *how much* one of them is higher or lower than another one. A method of index number calculation based solely on a given "system of tastes" as represented by a succession of indifference lines cannot possibly lead to any other result than a series of non-measurable magnitudes.[5] If, notwithstanding, these results are given in the form of definite numbers, we have to discard their numerical meaning entirely and take into account only the respective order of magnitudes. In so far as such an index number represents a *ratio* between two composite prices or quantities, its economic significance, if it exists at all, can be represented in terms of one of the three signs: >1, <1 or $=1$ (or, using percentages: >100 per cent, <100 per cent, $=100$ per cent). Any further numerical definiteness which an index number seems to convey is devoid of economic meaning. No wonder that every attempt toward a numerical

[5] It is interesting to note that Pareto called his indifference lines "*indices* of utility" stressing through the use of the word "index" their non-measurability.

interpretation will, in the given circumstances, produce nothing but confusion.

Applying these considerations to the interpretation of the numerical results obtained through application of the "method of limits," we see that the alleged "limits" are no limits at all. They do not indicate the numerical margins between which a "true" value is situated. All they can do is to show whether the magnitude of the "true" composite price or quality ratio is larger, equal to, or smaller than, one ($\gtreqless 1$).[6]

The following table indicates all the possible combinations of the magnitudes of the "upper" and "lower limits" and the corresponding magnitudes of the true *quantity index*.

TABLE 1

	"Upper limit" $\frac{\Sigma p_2 q_1}{\Sigma p_2 q_2}$	"Lower limit" $\frac{\Sigma p_1 q_1}{\Sigma p_1 q_2}$	The magnitudes of the "true" quantity index
I	1	1	1
II	1	>1	>1
III	1	<1	<1
IV	>1	1	>1
V	>1	>1	>1
VI	>1	<1	$\gtrless 1$ (indeterminate)
VII	<1	1	<1
VIII	<1	>1	$\gtrless 1$ (indeterminate)
IX	<1	<1	<1

This survey shows that the result obtained has an economic meaning only if both "limits" are larger or smaller than 1 and also in the singular cases where one or both of them equal 1: It signifies that the first point, as compared with the second, is situated on a higher, lower, or on the same indifference curve, respectively. In all other cases the result is purely negative; the magnitude of the true quality index remains entirely indeterminate.

[6] Given a series of *non-measurable magnitudes*, $a_1 < a_2 < a_3 < a_4 < a_5$, etc., it is, indeed, possible to indicate that one of them, say a_3, is located between some other two, say a_1 and a_4. But these limits must necessarily be expressed in terms of two magnitudes belonging to the same series. Dividing all the members of the original series, for example, by a_2, another succession of relative numbers can be obtained, $a/a_2 < 1 < a_3/a_2 < a_4/a_2 < a_5/a_2$ etc. This new series is more determined than the first one in so far as it is possible to tell whether any given element is larger than, smaller than, or equal to, 1. The statement that a_3/a_2 has as its limits the numbers 5/4 and 9/8 has no more meaning and conveys no more information than an indication that $a_3/a_2 < 1$.

In application to *price comparison*, the economic significance of the result obtained by application of the method of limits is even more restricted than in the case of quantity indices. The "limits" of a price-index, $\Sigma p_1q_1/\Sigma p_2q_1$ and $\Sigma p_1q_2/\Sigma p_2q_2$, are obtained by dividing the quotient of the total expenditures Σp_1q_1 and Σp_2q_2 in two positions by the two "limiting" quantity indices, $\Sigma p_1q_1/\Sigma p_1q_2$ and $\Sigma p_2q_1/\Sigma p_2q_2$:

$$\text{"lower limit" of the price index: } \frac{\Sigma p_1q_1}{\Sigma p_2q_2} : \frac{\Sigma p_1q_1}{\Sigma p_1q_2} = \frac{\Sigma p_1q_2}{\Sigma p_2q_2};$$

$$\text{"upper limit" of the price index: } \frac{\Sigma p_1q_1}{\Sigma p_2q_2} : \frac{\Sigma p_2q_1}{\Sigma p_2q_2} = \frac{\Sigma p_1q_1}{\Sigma p_2q_1}.$$

The total expenditures are *measurable* economic quantities and so is their quotient, $\Sigma p_1q_1/\Sigma p_2q_2$. The two "limits" of the *quantity* index, although expressed in numbers, have (with the exception of case 1 of Table 1) no definite numerical meaning. As shown above, in a number of cases they are void of any economic significance at all. It follows that we have to discard at the outset all the price indices calculated on the basis of such meaningless numbers. It stands to reason that any attempt to achieve a price comparison in the case where the composite quantities are entirely indeterminate would necessarily violate the very foundation of the whole procedure and lead to contradictory results. But the logical limits of possible price comparison, at least so long as it is based upon the "method of limits," are even much narrower than that.

The "true" quantity index, even if successfully obtained, in general has still no definite numerical meaning. It is a magnitude defined solely in its relation to unity.

The ratio of the numerical quotient of the expenditures and of the non-measurable magnitude of the quantity index cannot, in general, be anything else but a numerically indefinite magnitude itself, defined only through the sign $\gtrless \Sigma p_1q_1/\Sigma p_2q_2$:

$$\Sigma p_1q_1/\Sigma p_2q_2 : \lesseqgtr 1 = \gtrless \Sigma p_1q_1/\Sigma p_2q_2.$$

The following table covers all possible combinations of the sizes of the magnitudes involved in a price index computation.

The price index obviously remains entirely indeterminate if the magnitude of the corresponding quantity index is unknown (Case I) and, on the other hand, it aquires a definite numerical value in the singular situation in which the quantity index equals 1 (Case II). Otherwise, the expenditure ratio, $\Sigma p_1q_1/\Sigma p_2q_2$, indicates the lower (Case III) or the upper (Case IV) limit of the "true" price index (in the same way as unity can be called the limiting value of the "true" quantity index. See Table 1).

The economic significance of the latter result depends on the numeri-

cal value of the given expenditure ratio, $\Sigma p_1q_1/\Sigma p_2q_2$. If the magnitude of the price index is defined as in Case III ($> \Sigma p_1q_1/\Sigma p_2q_2$) and the expenditure ratio is at the same time larger than 1, say 1.5 (150 per cent), the price level in the first point of comparison is proved to be definitely higher then it is in the second point. However, should the expenditure ratio be smaller then 1, say 0.5 (50 per cent), the answer to the fundamental question concerning the *direction of the price level change remains entirely indeterminate,* because a magnitude defined through the sign >0.5 may be smaller, as well as larger, than 1. A similar indeterminateness must inevitably appear in Case IV if the expenditure ratio happens to be larger than 1.

Table 2 does not contain any reference to the previously mentioned "upper" and "lower limits" ($\Sigma p_1q_2/\Sigma p_2q_2$ and $\Sigma p_1q_1/\Sigma p_2q_1$) of the price index. The reason for this omission is the fact that the knowledge of these two numerical values cannot convey any information at all concerning the magnitude of the "true" price ratio. The preceding discussion makes it obvious that the indeterminate situation in Case I of Table 2 may coincide with every possible combination of magnitudes of the alleged "limit" of the price index. This means that, whatever their numerical values are, they not only do not convey any information concerning the direction of the "true" price change, but can not even indicate whether it is at all determinable on the basis of given statistical information.[7]

[7] A good example of such a difficulty in the interpretation of statistical data may be found on p. 46 (Table VII) of Dr. Staehle's book (*op. cit.*). A comparison of German (Berlin) and Danish (Copenhagen) family budgets gives a following series of values:

$$\Sigma p_1q_1 = \text{R.M. } 369.83, \qquad \Sigma p_1q_2 = \text{R.M. } 379.52,$$
$$\Sigma p_2q_2 = \text{R.M. } 334.77, \qquad \Sigma p_2q_1 = \text{R.M. } 337.94.$$

Deriving the two "limits" of the quantity index, we obtain:

$$\frac{\Sigma p_1q_1}{\Sigma p_1q_2} = \frac{369.83}{379.52} > 1,$$
$$\frac{\Sigma p_2q_1}{\Sigma p_2q_2} = \frac{337.94}{334.77} < 1.$$

According to Table I, this result shows that the "true" quantity index is entirely indeterminate and that, consequently, no price level comparison between the two countries is possible. And still the alleged "upper" and "lower" limits of the conventional price index give a definite indication that the price level in Copenhagen is higher then in Berlin:

$$\frac{\Sigma p_1q_1}{\Sigma p_2q_1} = \frac{337.94}{369.83} = 0.9138,$$
$$\frac{\Sigma p_1q_2}{\Sigma p_2q_2} = \frac{337.94}{379.52} = 0.8820.$$

In terms of the given theoretical assumptions, this result has obviously no economic meaning.

Summing up the economic analysis of the "method of limits," it is interesting to note that the often mentioned symmetry between any given price-index and the corresponding quantity index is rather superficial. So far as the fundamental statistical set-up is concerned, there exists, so to speak, a definite priority of the quantity index. The price magnitude of the composite commodity may be derived only if the magnitude of its "quantity"[8] can be also determined. In all the cases, however, in which the size of the total expenditures moves in the same direction as the magnitude of the composite quantities, the direction of the corresponding composite price change is fundamentally indeterminate.

TABLE 2

	Expenditure ratio	Magnitude of the quantity index	Magnitude of the "true" price index
I	$\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$	$\gtrless 1$	$\lessgtr \frac{\Sigma p_1q_1}{\Sigma p_2\, q_2}$
II	$\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$	1	$\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$
III	$\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$	<1	$>\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$
IV	$\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$	>1	$<\frac{\Sigma p_1q_1}{\Sigma p_2q_2}$

The foregoing analysis is not intended to be a criticism of the use of conventional index numbers in general; many theoretical and corresponding practical questions, by their very nature, require the application of indices of this specific form. To this class of problem belong, for example, comparisons of the standards of living. Our objections are directed only against the uncritical use of these kinds of composite prices and quantities in certain parts of general theoretical analysis where they are inappropriate. The principal cause of this misplacement seems to be the unwarranted assumption that the nature of these composite commodities is the same, or at least nearly the same, as the nature of the individual goods, that they are subject to the same elementary economic laws as ordinary commodities. We have tried to show that this assumption is utterly misleading; the magnitudes of

[8] Strictly speaking, this magnitude should not be called "quantity" because it is essentially non-measurable. For want of a better expression, this obviously contradictory use of terms seems to be unavoidable.

these composite index commodities are non-measurable, while measurability constitutes the fundamental economic characteristic of ordinary commodities and their prices.

III

5. The question still remains, whether it is possible to establish some other methods of reduction of the number of variables in the analysis of an economic system by lumping together individual goods and building up composites which would still retain the fundamental properties[9] (among them the measurability, of the "simple" commodities) and, consequently, fit into the body of general theory.

One answer to this question points to the much debated problem of the measurability of utility. Without discussing the controversial issue whether or not utility can be considered as a measurable magnitude (the opponents of this assumption seem still to have a better case), we may accept for a moment the measurability hypothesis and consider the theoretical implications resulting from the introduction of this new dimension. Such an identification of the concept of a quantity of the composite commodity and a quantity of utility, natural as it seems to be, proves much less useful than would be expected. In terms of such a composite quantity it would be impossible to formulate the fundamental principle of diminishing utility. This principle expresses a certain quantitative relation between the total amount of a commodity and the utility of successively added units. Such an interrelation assumes the existence of two independent scales of measurement, a quantity scale and a utility scale. But a distinction between the utility and the quantity in our composite commodity becomes meaningless if the utility unit itself is used as a yardstick for measurement of quantities. A similar situation arises if an attempt is made to use the quantity of a finished commodity as an index of the total physical amount of cost goods which are applied to its production. Described in terms of such composite cost units, every production process will necessarily appear to be subject to the law of strictly proportional returns.

Moreover, the method of lumping the individual commodities on the basis of their utility (or productivity) is subject to another very serious limitation. Even the most ardent proponents of the theory of measurable utility do not maintain that it should be possible to select

[9] All the "tests" applied to the determination of the goodness of different index formulae are nothing else than a reformulation of these fundamental properties of "simple" commodities.

at will any group of commodities and assign to them a definite utility index. The index could be applied only, if at all, to the combination of all commodities entering the "budget" of a given consumer. Analogously, the production cost index described above can be calculated only for all the cost elements entering a given production process taken together. There is no sense in speaking of an *independent* total utility of a group of two or three consumers' goods so long as they constitute only a part of the total consumption; it is impossible to evaluate the productivity (it should be understood that it is the total and not the marginal productivity that is meant) of two or three factors of production if more agents concur in the same process.

Whatever the general importance of this kind of index measurement may be, it obviously would not solve our problem. A composite commodity constructed according to these prescriptions, although measurable in its magnitude, is subject to so many general limitations and displays at the same time such peculiar characteristics that it would be impossible to incorporate it in the body of the generally accepted economic theory *at par* (side by side) with all the other, non-composite goods. It seems to be advisable to attack our problem from a somewhat different angle.

6. To make the scope of this discussion more general, we change the previous scheme and, instead of two commodities (A and B) and money (M), start with a system of three commodities A, B, C; one of them may easily be replaced by money at any stage of the argument. The utility interrelations among the three goods can be described in a three-dimensional diagram by a series of indifference surfaces. Analogous to a series of indifference curves, a system of superimposed indifference *surfaces* gives a complete and theoretically precise description of the interrelations among three commodities. Each of the indifference surfaces will indicate all possible equivalent combinations of A, B, and C, i.e., as an indifference line represents a certain utility level. The relative positions of successive indifference surfaces will indicate "higher" and "lower" utility levels. Figure 5 gives a schematic representation of such an indifference surface (i-y-e).[10]

In terms of the graph, our index-problem amounts to a translation of the three-dimensional picture into a two-dimensional diagram. This new diagram must have exactly the same general properties as the previously analysed two-dimensional system of indifference curves. Fur-

[10] In the same way as an indifference line may approach the coordinates of a two-dimensional diagram without actually cutting them, an indifference surface may never cut the limiting planes AoB, AoC, and BoC.

thermore, the translation must be accomplished in such a way that while one of the axes will represent the quantities of the new combination of B and C—we will call it I—the other will still measure the original quantities of the third commodity, A.

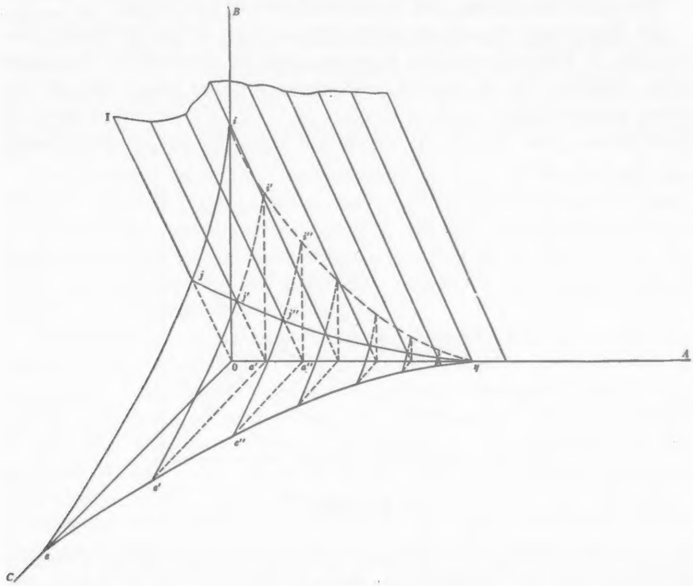

FIGURE 5

A solution of this problem is facilitated by the two following considerations:

(a) A simple measurement of the different quantities of the new composite commodity in terms of physical units will be possible only under the condition that its *composition remains constant.* Each unit of the composite good, I, must contain a fixed amount of B and C.

(b) In order to reduce the different combinations of B and C to this single denominator, this criterion can readily be applied: Any of this combination can be replaced by a definite equivalent quantity of I, which would mean an exactly equal amount of utility to the consumer.

In terms of the graph (Figure 5), a solution of our problem consists in putting a plane IoA through the original system of indifference surfaces, the plane footing on the abscissa, oA, and inclined at a cer-

tain deliberately selected angle toward the two other coordinates. The cross cuts between this plane and the successive indifference surfaces (on Figure 5 only one such surface is shown) produce a series of curves and represent the desired new set of indifference lines.

The whole transformation is accomplished in two steps:

(a) The first is the choice of the composition of the synthetic commodity, I. This composition is represented by the slope of the plane IoA. Cutting the system of indifference surfaces along BoC or any other parallel plane, such as $i'\ a'\ e'$ or $i''\ a''\ e''$, we get a series of indifference lines (Figure 6) which will represent the utility relation

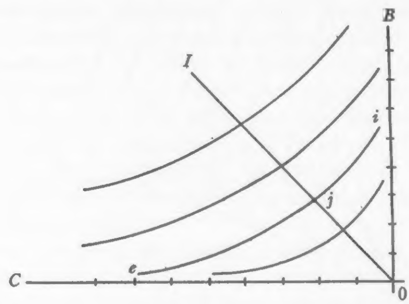

FIGURE 6

between B and C on the assumption of a constant amount of A (which is equal to o, a', a'', respectively, for the crosscuts mentioned above). The inclination of the plane IoA will be equal to the slope of the line ojI. This slope corresponds to a fixed relation between the quantities of B and C which we choose to combine in each unit of the new composite commodity. This proportion may be e.g., $1B$ to $2C$, $1B$ to $1C$, or any other. Theoretically, the choice of this proportion is absolutely free. Once the composition of the composite commodity is thus established, i.e., the slope of ojI is fixed, its quantity can be measured along the line from o toward I in the same manner as the amounts of A, B, and C, are measured along their separate coordinates.

(c) The amount of Oj of I is equivalent in its utility to all the other combinations of B and C which are situated on the indifference line $i\ j\ e$. Similarly, all the points of the indifference curves $i'\ j'\ e'$, $i''\ j''\ e''$, etc. (Figure 5), are equivalent to the corresponding amounts of I: $a'\ j'$, $a''\ j''$, etc. Furthermore, we see that the points j', j, j'', y, being all situated on the same indifference surface, $i\ e\ y$, necessarily represent also an equal utility level. The curve $j\ j'\ j''\ y$ (Figure 4) represents an

indifference line and describes the utility relation between the composite commodity, I, and the good, A. The three-dimensional utility surface, $j\ e\ y$, is reduced to a two-dimensional indifference curve, $j\ y$. In the same way all the other utility levels of the original three-dimensional system can be represented by corresponding indifference curves on the plane IoA.

With the help of a similar transformation, any number of individual commodities can be reduced to a single composite good. Once the substitution is accomplished, this new abstract good can be treated theoretically exactly in the same way as if it were a simple element of the initial uncondensed system.

If the price level of the two commodities must be determined, the problem consists in translating each of the separate points of comparison to the common denominator I, the previously selected composite commodity. The price levels are obtained by division of the total expenditures in the particular points by the corresponding amounts of the composite commodity.

An element of *arbitrariness* enters the described procedure with the choice of the "composition" of the index commodity (the slope of the cross cut $o\ J$ on Figures 5 and 6). As already mentioned, theoretically any slope between $o\ B$ and $o\ C$ will serve the purpose. But it is difficult to see why this circumstance should impair the usefulness of the method. Whatever the choice of the composite commodity, the index will always fulfill the fundamental requirements noted and consequently serve its purpose, which is to the simplification of the system through reduction of the number of variables. The results obtained on the basis of each of the different index commodities will necessarily be compatible with each other and in no case contradictory. Moreover, the transition from one composition of the index commodity to a different one can be accomplished by means of the same technique—translation along the indifference curves.

For index calculations it seems to be advisable to choose the slope of the index cross cut in such a way as to reduce as much as possible the distances between the original combination of the given commodities and the corresponding index points. It would be wrong, however, to use for this purpose a definite formula, to compose, for example, the index commodity according to the average composition of all empirically given original points. A rigid arrangement of this sort would necessitate a revision of the whole system every time a new point was added to the original group of data, which is a complication warranted neither by theoretical nor by practical considerations.

7. The statistical usefulness of this proposed method depends mainly on the possibility of discovering the shape of the indifference curves—

an empirical task the difficulty of which can hardly be exaggerated. The previous discussion showed, however, that a given set of indifference curves—although not always explicitly mentioned—is the common basis of many of the rival theoretical approaches to the problem of composite commodities. Because of this, the practical difficulty cited above affects equally all of these theories. Analysis of the usual statistical technique shows that the slightest neglect of this "system of tastes" must in any case lead to faulty results.

As things are, a preliminary practical solution must be sought in deliberate acceptance of some more or less warranted approximations. But, in this respect also, none of the different theoretical approaches could possibly claim a practical advantage.

In connection with this contention it is interesting to see that, with the help of certain "heroic" assumptions, it is even possible to interpret the numerical magnitudes obtained by the "method of limits" in terms of the proposed composite commodities.

Given two points Q_1 and Q_2 (Figure 2) and the two corresponding price ratios. The latter are represented by the slopes of the two lines, *cb* and *ad*. If we now select the proportional combination of the two commodities, A and B, in point Q_1 as our standard combination, the line otQ_1v will represent the direction along which the quantity of the composite commodity has to be measured. According to this set-up, the distance oQ_1 will represent the composite amount in point Q_1. A theoretically correct procedure would require a translation of the point Q_2 along the indifference line $i-i$, and the result obtained would indicate that the combination Q_2 represents exactly the same amount (oQ_1) of the composite commodity as combination Q_1, which means that the quantity index is equal to 1. If the shape of the indifference line is unknown, it is still possible to attempt some kind of approximation. From the fundamental properties of the indifference lines, it follows that the crosspoint between the line otQ_1v and the presumably unknown indifference line on which the point Q_1 is located can in no case be lower than t or higher than v. In other words, the unknown quantity index will certainly not be larger than oQ_1/ot and not smaller than oQ_1/ov. These two ratios may be called the upper and lower limit of the quantity index calculated on the basis of the composite commodity otQ_1v. From the given geometric relations it follows that

$$oQ_1/ot = Ko/ao = \Sigma p_2q_1/\Sigma p_2q_2.$$

The latter expression is identical with the "lower limit" of the usual method of limits. The other ratio, oQ_1/ov, has no such well known counter part. In order to arrive at the ratio $\Sigma p_1q_1/\Sigma p_1q_2$ (the conven-

tional "upper limit" of the quantity index), it is necessary to *change the composition of the standard commodity* and perform the measurement along orQ_2w instead of otQ_1v. Applying the same reasoning as before, we find that, according to this new scale, the "true" quantity index must lie between ow/oQ_2 and or/oQ_2. And now,

$$\frac{or}{oQ_2} = \frac{oC}{oM} = \frac{\Sigma p_1 q_1}{\Sigma p_1 q_2},$$

i.e., the "upper limit" of the conventional method is identical with the upper limit calculated along the new scale. The conventional comparison of the two "limits," $\Sigma p_1 q_1 / \Sigma p_1 q_2$ and $\Sigma p_2 q_1 / \Sigma p_2 q_2$, appears to be rather illogical.

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A NOTE ON DISTRIBUTION OF INCOME OVER TIME[1]

By Gerhard Tintner

In the following note an attempt will be made to use the technique of the calculus of variations, which has proved itself so useful elsewhere in economic theory,[2] for the treatment of the following problem: Suppose the utility of a certain quantity of goods to depend not only on the amount possessed at any moment but also on the flow of goods in time. This can be fairly realistically interpreted in the case of consumers' goods. There may be commodities to which the individual in question may get more and more accustomed and the utility of which increases more rapidly the more the commodities have been consumed in the past. This may be true in the case of many individuals for certain luxury goods and services. The ordinary case, however, will be that the individual in question strives for change and novelty: the utility of a given amount of goods will be less the more has been consumed in the past.

The technique adopted here will be very similar to the treatment of demand functions with limited budgets by Professor Hotelling.[3] We will first follow his line of argument and treat the case of optimum utility without regard to time.

Let ϕ be a utility function which is supposed to be known and utility itself be supposed to be measurable.[4] $q_1 \cdots q_n$ may be the quantities of n goods and $p_1 \cdots p_n$ their prices. m denotes the income in money and is defined as

$$m = \sum_i p_i q_i \qquad (i = 1, 2 \cdots n). \tag{1}$$

[1] The author is much indebted to Professors Joseph A. Schumpeter and Harold Hotelling for advice and improvement on this paper. Mr. Paul Sweezy was kind enough to correct the English.

[2] G. C. Evans, *Mathematical Introduction to Economics*, New York, 1930. C. F. Roos, *Dynamic Economics*, Bloomington, Ind., 1934.

[3] H. Hotelling, "Demand Functions with Limited Budget," Econometrica III, Jan. 1935, 66 ff. See also Walras, *Elements d'Économie Politique Pure*, 4 ed., Paris, 1900, pp. 73 ff.

[4] Utility is here conceived as a pure concept with no relation to reality whatever. It enables us, however, to state the economic problems of the pricing process consistently and systematically and is very useful from this point of view. The integration of utility over time, which is the main subject of the present paper, only permits linear transformations if, as in the analysis of Pareto, some index function should be substituted for the utility function. Every other transformation would change the disposition of goods over time.

The problem is to maximize the function ϕ subject to the condition (1). This can be very elegantly solved by the use of a Lagrange multiplier λ, as Hotelling has shown. The function F to be maximized will be

$$F = \phi + \lambda \sum_i p_i q_i. \tag{2}$$

This at once leads to the n equations,

$$\frac{\partial F}{\partial q_i} = \frac{\partial \phi}{\partial q_i} + \lambda p_i = 0 \qquad (i = 1 \cdots n), \tag{3}$$

which, together with the equation (1), are sufficient for the determination of the $n+1$ unknowns $q_1 \cdots q_n$ λ, and the problem is solved.[5] The solution, however, can be represented in a somewhat different way.

Multiplying (3) by q_i and summing with respect to i, we get

$$\sum_i q_i \frac{\partial \phi}{\partial q_i} + \lambda \sum_i p_i q_i = 0, \tag{4}$$

and the Lagrange multiplier λ (the marginal utility of money) as

$$\lambda = -\frac{\sum_i q_i \frac{\partial \phi}{\partial q_i}}{\sum_i p_i q_i}. \tag{5}$$

Inserting this value in (3), we get the following expression for the weighted marginal utility (ophelimité ponderée):[6]

$$\frac{\partial \phi}{\partial q_i} \frac{1}{p_i} = \frac{\sum_i q_i \frac{\partial \phi}{\partial q_i}}{\sum_i p_i q_i} \qquad (i = 1 \cdots n). \tag{6}$$

The expressions in (6) have the following meaning: The weighted marginal utility $\dfrac{\partial \phi}{\partial q_i} / p_i$ is the marginal utility divided by the price, or the marginal utility, not of the unit of the good, but of the quantity which the unit of money will buy. The expression $\sum_i p_i q_i$ has been de-

[5] H. Hotelling, *loc. cit.*, p. 71.

[6] V. Pareto, *Manuel d'Économie Politique*, Paris, 1909, pp. 159 ff.

fined in (1) as the money income. The expression $\sum_i q_i \dfrac{\partial\phi}{\partial q_i}$ is the sum of quantities $q_i \dfrac{\partial\phi}{\partial q_i}\cdot$ Those quantities are nothing else than the value of a stock q_i of the good, in terms of utilities, according to Wieser.[7] So the meaning of the sum is the sum of values of the goods possessed.

Dividing the right-hand side of (6) by $\sum_i q_i$ in the denominator and numerator, we get

$$\frac{\partial\phi}{\partial q_i}\frac{1}{p_i} = \frac{\sum_i q_i \dfrac{\partial\phi}{\partial q_i}}{\sum_i q_i} \Bigg/ \frac{\sum_i p_i q_i}{\sum_i q_i} \qquad (i = 1 \cdots n). \tag{7}$$

The right-hand side of this expression can be interpreted in the following way: $\sum_i q_i \dfrac{\partial\phi}{\partial q_i} \Big/ \sum_i q_i$ is the weighted average of the marginal utilities, with the quantities as weights; and $\sum_i p_i q_i / \sum_i q_i$ is a kind of price index, the weighted average of prices with the quantities as weights. So the weighted marginal utility turns out to be the relation between the weighted average of marginal utilities and the weighted average of prices.

Multiplying (6) by q_i, and rearranging in the form of a proportion, we get the following result:

$$q_i \frac{\partial\phi}{\partial q_i} : \sum_i q_i \frac{\partial\phi}{\partial q_i} = q_i p_i : \sum_i q_i p_i \qquad (i = 1 \cdots n). \tag{8}$$

Here $q_i \dfrac{\partial\phi}{\partial q_i}$ stands for the value of the stock of the good in question: $\sum_i q_i \dfrac{\partial\phi}{\partial q_i}$ is the sum of the values of all stocks of goods, $q_i p_i$ is the sum of money outlay for the particular commodity, and $\sum_i q_i p_i$ is the

[7] F. v. Wieser, "Theorie der gesellschaftlichen Wirtschaft," in *Grundriss der Sozialoekonomik*, vol. I/2, 2 ed., Tuebingen, 1924, p. 70 f. H. Mayer, "Untersuchungen zum Grundgesetz der wirtschaftlichen Wertrechnung," *Zeitschrift fuer Volkswirtschaft und Sozialpolitik*, Neue Folge, Vienna, I, 1921, 431 ff., II, 1922, 1 ff.

total money income. From this we conclude that the value of the stock of every good must be in the same relation to the sum of the values of all stocks of goods as the sum of money paid for the quantity of this particular good to the total money income. A slight rearrangement gives us

$$(9) \qquad q_i \frac{\partial \phi}{\partial q_i} : q_i p_i = \sum_i q_i \frac{\partial \phi}{\partial q_i} : \sum_i p_i q_i \qquad (i = 1 \cdots n),$$

which means the following: The relation of the value of the stock or quantity of the particular good to the sum of money spent on it must be the same as the relation of the sum of values of the stocks of all goods to the total money income.

Now we come to our problem of the distribution of income over time. We define another utility function ψ which is supposed to depend not only on the quantities $q_1 \cdots q_n$ of the n goods but also on their flow in time, $q_1' \cdots q_n'$. The prices $p_i \cdots p_n$ are supposed to be fixed.

The problem is now to maximize the utility over a given interval of time, $t_0 t_1$, under the restriction that the income is given by

$$(10) \qquad M = \sum_i p_i \int_{t_0}^{t_1} q_i dt.$$

This can be formulated as a problem of a restricted maximum of an integral and solved by the introduction of a Lagrange multiplier μ:

$$(11) \qquad \int_{t_0}^{t_1} G dt = \int_{t_0}^{t_1} \left(\psi + \mu \sum_i p_i q_i \right) dt.$$

The function G which is to be maximized must satisfy the so-called Euler equations as a necessary condition:

$$(12) \qquad \frac{\partial G}{\partial q_i} - \frac{d}{dt} \frac{\partial G}{\partial q_i'} = 0 \qquad (i = 1 \cdots n).$$

This leads in our case to the following differential equations:

$$(13) \qquad \frac{\partial \psi}{\partial q_i} + \mu p_i - \frac{d}{dt} \frac{\partial \psi}{\partial q_i'} = 0 \qquad (i = 1 \cdots n).$$

The number of such differential equations is n and, together with equation (10), they give solutions to the $n+1$ unknowns $q_1 \cdots q_n$ μ and determine their development in time, if another condition is fulfilled. The equations (13) are partial differential equations of the second order and their solution involves $2n$ constants of integration, which

also must be given, for instance by fixing the value of the q_i for t_0 and t_1.

We can, however, transform our differential equations (13) in a way similiar to the case previously considered. Multiplying through by q_i and summing over i, we get the following expression:

$$\sum_i q_i \frac{\partial\psi}{\partial q_i} + \mu \sum_i p_i q_i - \sum_i q_i \frac{d}{dt} \frac{\partial\psi}{\partial q_i'} = 0. \tag{14}$$

This gives us the Lagrange multipliers μ as:

$$\mu = \sum_i q_i \left(\frac{d}{dt} \frac{\partial\psi}{\partial q_i'} - \frac{\partial\psi}{\partial q_i} \right) \Big/ \sum_i p_i p_i, \tag{15}$$

and (13) can be written in a form similar to (6):

$$\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt} \frac{\partial\psi}{\partial q_i'} \right) \frac{1}{p_i} = \frac{\sum_i q_i \left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt} \frac{\partial\psi}{\partial q_i'} \right)}{\sum_i p_i q_i} \quad (i = 1 \cdots n). \tag{15}$$

The interpretation of (15) is not very easy. We see at once that the whole expression degenerates into equation (6) of the simple weighted marginal utility if we assume ψ to be independent of the rate of change of the quantities of goods q_i'. The left-hand side of (15) is a kind of weighted marginal utility itself. It is, moreover, the form of the Euler equations if the function ψ has an unrestricted maximum and in this case is equal to zero. The expression in brackets on the left-hand side is the difference between the marginal utility of the good in question and the variation with respect to time of the quantity $\partial\psi/\partial q_i'$, which is a kind of marginal utility in the sense that it expresses the influence of the flow of the good in time on the utility function. We have tried to give an economic meaning to an expression like this at the very beginning of the argument. The right-hand side of equation (15) contains in the denominator the sum of expressions such as the one just discussed for all goods and in the numerator the amount of money spent at the particular moment.

If we divide on the right-hand side in the numerator and denominator of (15) by $\sum_i q_i$, we get an expression similar to (7):

$$\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt} \frac{\partial\psi}{\partial q_i'} \right) \frac{1}{p_i} = \frac{\sum_i q_i \left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt} \frac{\partial\psi}{\partial q_i'} \right)}{\sum_i q_i} \Bigg/ \frac{\sum_i p_i q_i}{\sum_i q_i}. \tag{16}$$

$$(i = 1 \cdots n).$$

The expression on the left-hand side, the weighted difference between the marginal utility of the good in question and the variation in time of the marginal influence of its rate of change in time, equals again the ratio of two weighted averages: In the numerator we have the weighted sum of expressions like the bracketed part of (15) with the quantities consumed in this moment as weights, and in the denominator stands the average of prices with the quantities of goods as weights.

It is not too difficult to derive the proportion analogous to (8),

$$(17)\quad q_i\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt}\frac{\partial\psi}{\partial q_i'}\right) : \sum_i q_i\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt}\frac{\partial\psi}{\partial q_i'}\right) = p_iq_i : \sum_i p_iq_i \qquad (i = 1 \cdots n).$$

The first expression is the quantity of the good multiplied by the difference between the marginal utility and the derivative with respect to time of the marginal influence of the rate of change in time of this commodity on the utility function. The next expression is the sum of such quantities for all goods. The ratio of those two magnitudes must be the same as the amount of money spent in the moment for the particular good to the total amount of money spent in this moment on all the goods.

The proportion analogous to (9) is

$$(18)\quad q_i\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt}\frac{\partial\psi}{\partial q_i'}\right) : q_ip_i = \sum_i q_i\left(\frac{\partial\psi}{\partial q_i} - \frac{d}{dt}\frac{\partial\psi}{\partial q_i'}\right) : \sum_i p_iq_i \qquad (i = 1 \cdots n),$$

and its interpretation is analogous to the foregoing. We see that the solution of the problem of the distribution of income over time has interesting formal analogies to the general and simpler problem of distribution of income on various goods which has been dealt with so admirably by Hotelling. We may add that the formalism of the solutions would be a very similar one if we considered not only the influence of the first but also of higher derivatives of the q_i with respect to time on the utility function.

Returning to equation (12), we can reach another solution of the Euler equations which may prove useful for the economic interpretation. Since the function G defined by (11) does not explicitly include t, we can write the first integral of (12) in the following manner:

$$(19)\qquad G - q_i'\frac{\partial G}{\partial q_i'} = \psi_0 \qquad (i = 1 \cdots n),$$

where ψ_0 is a constant of integration. Inserting the expression for G from (11), we get

$$\psi + \mu \sum_i p_i q_i - q_i' \frac{\partial \psi}{\partial q_i'} = \psi_0 \qquad (i = 1 \cdots n). \tag{20}$$

The value of μ from (15) can be written in a slightly different form,

$$\mu = - \sum_i \frac{q_i}{q_i'} \frac{d}{dt}\left(\psi - q_i' \frac{\partial \psi}{\partial q_i'}\right) \Big/ \sum_i p_i q_i. \tag{21}$$

Inserting this value in (20), we get the following simple expression for the first integral of the Euler equations:

$$\psi - q_i' \frac{\partial \psi}{\partial q_i'} = \sum_i \frac{q_i}{q_i'} \frac{d}{dt}\left(\psi - q_i' \frac{\partial \psi}{\partial q_i'}\right) + \psi_0 \qquad (i = 1 \cdots n). \tag{22}$$

We see at once the meaning of the constant of integration, ψ_0. If the utility function were independent of the rate of flow of the goods, the total utility derived ψ would equal ψ_0. The left-hand side is the difference between the total utility derived in the point of time and the marginal influence of the flow of the good in time multiplied by the rate of change in time. The rest is the sum of the rate of changes in time of such expressions for all the goods, with the quantities as weights.

The n equations (22) again lead us to a perfect determination of the n unknowns q_i as functions of time and in this way are a solution of the problem of the distribution of income over time.

We should not forget, however, to mention that we only stated the necessary and not the sufficient conditions of a maximum. The latter have been developed fully by Hotelling.[8] Second order conditions in the calculus of variations, however, are fairly complicated. If stated, they would enable us to learn more about the structure of the utility function, which must satisfy certain inequalities in order that the problem of the distribution of income over time should have a determinate solution.

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[8] H. Hotelling, *loc. cit.*, pp. 71 ff.

MATHEMATICAL THEORY OF PRODUCTION STAGES IN ECONOMICS[1]

By JOHN M. THOMPSON

IN the first part of this paper, we distinguish between length of production in terms of time and length of production in terms of stages. The relationship between the length of production in terms of stages and the ratio of the monetary demand for producers' goods to the monetary demand for consumers' goods is found to depend on the difference, voluntary savings minus loans to consumers. It is shown that when total money payments are constant, the length of production varies directly with voluntary savings and oppositely to income and loans to consumers. We also present an answer to the question regarding the permanence of a change in the length of production due, first, to voluntary savings and, second, to loans to producers. Finally, one point of disagreement between Dr. F. A. Hayek and Drs. Hansen and Tout is discussed. We hope that this mathematical formulation of the length of production will aid in refining our use of this concept.

1. *Introduction.*—The concept, length of production or number of production stages, is fundamental to some recent theories of the trade cycle. Hence, it is desirable to give a mathematical formulation of this notion as an aid in determining definitions, hypotheses, and deductions, appropriate to its use. Foremost in the description of this concept and its application to monetary theory is Dr. F. A. Hayek and, accordingly, this paper will have special regard to his book, *Prices and Production.*[2]

2. *Definitions.*—The following definitions may all apply to a single commodity among many or to a composite commodity. If to a single commodity, a general length of production for the whole society must be some weighted mean of the lengths of production for each commodity.

2.1. The symbols, $G_p(t)$, $G_c(t)$, are the amounts of production goods and consumption goods produced per unit time. The respective demands for production goods and consumption goods per unit time expressed in terms of money are represented by $D_p(t)$, $D_c(t)$; and $p_p(t)$, $p_c(t)$, are the corresponding prices.

[1] The possibility of formulating a mathematical theory of production stages was suggested by Prof. Griffith C. Evans.

[2] F. A. Hayek, *Prices and Production*, London, 1931. Cited hereafter as Hayek.

2.2. By uniform production it is meant that $G_c(t) = \text{constant}$.

2.3. Coefficient of money transactions is $C_{MT}(t) = \dfrac{G_M(t)}{G(t)}$, where $G_M(t)$ is the amount of goods exchanged against money per unit time and $G(t)$ is the total "flow" of goods (whether exchanged against money or not) per unit time. In symbols,

$$G_M(t) = \frac{p_p(t)G_p(t) + p_c(t)G_c(t)}{p_M(t)},$$

where $p_M(t)$ is some average price of all goods exchanged against money. An accurate determination of $G(t)$ is impossible, but its amount can be determined roughly in terms of index numbers, for a transaction without money in one firm or industry is being made with money in another.[3] The coefficient of money transactions would decrease with an increase in barter or the vertical integration of firms; it would increase with a decrease in barter or disintegration of firms. In terms of differentials,

$$dC_{MT}(t) = \frac{G(t)dG_M(t) - G_M(t)dG(t)}{[G(t)]^2}.$$

2.4. Neutral money means $M(t)\,V(t) + M'(t)\,V'(t) = \text{constant}$, except for changes necessary to counteract variations in $C_{MT}(t)$. The symbol, $M(t)$, is the amount of currency and $M'(t)$ is the amount of bank deposits subject to check; $V(t)$, $V'(t)$, are their respective velocities of circulation.

2.5. Voluntary savings invested per unit time in money is represented by $E(t)$.

2.6. Net money borrowed per unit time by entrepreneurs is represented by $F_p(t)$; and $F_c(t)$ is the net money borrowed per unit time by consumers. The word "net" refers to the difference between loans and repayment of loans per unit time. Hence, $F_p(t)$ and $F_c(t)$ may be positive, negative, or zero.

2.7. Consider the production units (e.g., automobiles) as passing through the stages (moulding the block for the motor, or painting the body, etc.) of the production process as time varies. The total labor put in the i-th production unit from time $t=0$ to $t=t$ is $l_i(t)$

[3] Since a large part of $G(t)$ is not exchanged against money, a coefficient of "monetization" that could be determined more accurately is

$$\frac{1}{G_M(t)}\frac{dG_M(t)}{dt} = \frac{d}{dt}\log G_M(t).$$

$= \int_0^t \dot{l}_i(t)dt = \int_0^t dl_i(t)$. The total land put in the i-th production unit from time $t=0$ to $t=t$ is $s_i(t) = \int_0^t \dot{s}_i(t)dt = \int_0^t ds_i(t)$.

For simplicity, only two original factors of production are considered, but labor may be considered as including all types of human services entering the production process and land may include all natural resources.

2.8. The fundamental concept, length of the structure of production, may be thought of as the average length of time the original factors of production are invested in the production unit (more accurately, average length of time a unit of value of the original factors of production is invested). In symbols,[4]

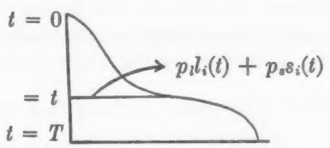
$$L_i(T) = \frac{\int_0^T \{p_{l_i}(t)\dot{l}_i(t) + p_{s_i}(t)\dot{s}_i(t)\}[T - t]dt}{\int_0^T \{p_{l_i}(t)\dot{l}_i(t) + p_{s_i}(t)\dot{s}_i(t)\}dt},$$

where T is the total length of time from the first entrance of land and labor in the i-th production unit until it is finished, and $p_{l_i}(t)$, $p_{s_i}(t)$, are the prices of labor and land entering the production unit at time t.

If $L_i(T)$ is expressed as a Stieltjes Integral and $p_{s_i}(t) = p_s$, $p_{l_i}(t) = p_l$ constants, then it can be put in a different form:

$$L_i(T) = \frac{\int_0^T p_{l_i}(t)[T - t]dl_i(t) + \int_0^T p_{s_i}(t)[T - t]ds_i(t)}{\int_0^T \{p_{l_i}(t)dl_i(t) + p_{s_i}(t)ds_i(t)\}},$$

$$= \frac{p_l \int_0^T l_i(t)dt + p_s \int_0^T s_i(t)dt}{p_l l_i(T) + p_s s_i(T)}.$$



If $p_{l_i}(t)$, $p_{s_i}(t)$, $\dot{l}_i(t)$, $\dot{s}_i(t)$, are constants (Hayek's hypotheses), then

$$L_i(T) = \frac{\int_0^T [T - t]dt}{\int_0^T dt} = \frac{T}{2}.$$

2.9. In much of his work, Dr. Hayek considers the length of production equal to the ratio of the total monetary demand for producers' goods to the total monetary demand for consumers' goods.

[4] C. H. P. Gifford has given a similar definition in his article "The Concept of the Length of the Period of Production," in the *Economic Journal*, XLIII, 1933, 612.

The preceding definition of the length of production has the dimension one in time[5] and, therefore, cannot be equal to the dimensionless expression, $\frac{D_p(t)}{D_c(t)}$. Accordingly, although the above concept is the one which Hayek seems to intend, it cannot be the one which he uses in his theorems.

It is possible, however, to devise a concept which is still in line with Hayek's ideas and fits them perhaps even more closely than his own words. We shall, therefore, construct a definition of length of production which is of zero dimension in all fundamental units and is equal to $\frac{D_p(t)}{D_c(t)}$ under special conditions. This second definition is in terms of stages, that is, *the length of production is the average number of stages that land and labor pass through after entering production.*

The value of land and labor entering the q-th stage during time t_{q-1} to t_q on the production units beginning at t_0 is

$$\int_{t_{q-1}}^{t_q} \{p_l{}^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t) + p_s{}^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt,$$

$$(t_{q-1} = 0 \text{ for } q = 1)$$

where $p_l{}^{(q)}(t_0+t)$ and $p_s{}^{(q)}(t_0+t)$ are the prices of labor and land entering the q-th stage at time (t_0+t), $\dot{l}^{(q)}(t_0+t)$ and $\dot{s}^{(q)}(t_0+t)$ are the amounts of labor and land entering the q-th stage per unit time at (t_0+t).

The value of labor and land entering the q-th stage on the production units beginning at t_0, multiplied by the number of stages this value passes through, is

$$(\tau - q + 1)\int_{t_{q-1}}^{t_q} \{p_l{}^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t) + p_s{}^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt.$$

Summing this expression for all τ stages, we have

$$\sum_{q=1}^{\tau} (\tau-q+1)\int_{t_{q-1}}^{t_q} \{p_l{}^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s{}^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt.$$

The average number of stages through which the labor and land entering the production units beginning at t_0 pass, where the number of stages passed through is weighted according to the total value entering the q-th stage, is

[5] Evans, *Mathematical Introduction to Economics* New York 1930, Chapter 2.

$$L^{\tau}(t_0)=\frac{\sum_{q=1}^{\tau}(\tau-q+1)\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt}{\sum_{q=1}^{\tau}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt}.$$

The average number of stages that the labor and land entering the production units beginning during the time $(0\mathfrak{I})$ are in the process is

$$L^{\tau}(\mathfrak{I})=\frac{\int_0^{\mathfrak{I}}\left[\sum_{q=1}^{\tau}(\tau-q+1)\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt\right]dt_0}{\int_0^{\mathfrak{I}}\left[\sum_{q=1}^{\tau}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt\right]dt_0}$$

If $p_l^{(q)}(t_0+t)$, $p_s^{(q)}(t_0+t)$, $\dot{l}^{(q)}(t_0+t)$, $\dot{s}^{(q)}(t_0+t)$, are constants, and (t_q-t_{q-1}) is the same for all stages, then

$$L^{\tau}(t_0)=\frac{\sum_{q=1}^{\tau}(\tau-q+1)}{\sum_{q=1}^{\tau}(1)}=\frac{\frac{\tau(\tau+1)}{2}}{\tau}=\frac{\tau}{2}+\frac{1}{2}, \text{ and}$$

$$L^{\tau}(\mathfrak{I})=\frac{\tau}{2}+\frac{1}{2}.$$

3. *Assumptions.*—The first three constraints on the system will remain throughout the discussion; but the last three, 3.4, 3.5, 3.6, assumed initially, are removed later.

3.1. Monetary value of the demand for production goods is equal to the monetary value of the supply of production goods, and similarly for consumption goods:[6]

$$D_p(t)\equiv p_p(t)G_p(t);\qquad D_c(t)\equiv p_c(t)G_c(t).$$

3.2. There is no hoarding:[7]

$$D_c(t)\equiv I(t)-E(t)+F_c(t).$$

Here $I(t)$ denotes the quantity $\sum_{q=1}^{\tau}\{p_l^{(q)}(t)\dot{l}^{(q)}(t)+p_s^{(q)}(t)\dot{s}^{(q)}(t)\}$. Although hoarding is an important element in trade cycle theory, our assumption of its non-existence leads to no loss of generality, since in this analysis a positive (negative) amount of net hoarding per unit time

[6] Hayek does not explicitly state this assumption, but Figure 2, page 40, and statements on page 42 show that he considers it so.

[7] This combines three assumptions made by Hayek on pages 41, 46, and 55.

has exactly the same influence as an equal but negative (positive) amount of $F_c(t)$.

3.3. The coefficient of money transactions is held constant (Hayek, p. 41).

3.4. Neutral money is assumed until section 6 (Hayek, p. 48).

3.5. The system is one of uniform production, that is, for the present we assume that the rate of production is constant in any given process (Hayek, p. 42).

3.6. We assume that $E(t)-F_c(t)\equiv 0$, until section 5.

3.7. Hayek's interest in the concept of length of production lies in the assumption of its relation to the capital character of production: that the amount of consumption goods produced increases or decreases with the length of production for a given amount of labor and land entering production.[8] This POSTULATE OF THE INDUSTRIAL RÉGIME may be stated in symbols as follows:

$$G_c(t) = G_c(L, \mathcal{L}(t), S(t)); \quad \frac{\partial G_c(t)}{\partial L} > 0,$$

$$\text{where } \mathcal{L}(t) = \sum_{q=1}^{r} \dot{l}^{(q)}(t), \text{ and } S(t) = \sum_{q=1}^{r} \dot{s}^{(q)}(t).$$

On the basis of this postulate, our problem is, therefore, restricted to a discussion of $L^r(t)$.

4. *Hayek's basic theorem.*—We wish to see what deductions can be drawn from these definitions and assumptions. By means of a double sum, we may express the total amount of money that is exchanged against the production units beginning at time t_0,

$$\sum_{r=1}^{r}\sum_{q=1}^{r}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt.$$

The total value of production goods developed on production units beginning during an interval of time $(0\mathfrak{T})$ is

$$\int_0^{\mathfrak{T}}\left[\sum_{r=1}^{r}\sum_{q=1}^{r}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}dt\right]dt_0.$$

The total value of production goods produced during the same time is

$$\int_0^{\mathfrak{T}} G_p(t)p_p(t)dt.$$

These two values may be considered equal if we have slow changes in our production system, that is, if the part of $G_p(t)p_p(t)$ near the end of the process while $0<t<T$ is balanced by the quantity

[8] Hayek, p. 34.

$$\sum_{r=1}^{\tau}\sum_{q=1}^{\tau}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt,$$

when $\mathfrak{Z}<t<\mathfrak{Z}+T$. This statement is expressed as follows:

$$\text{(A)}\quad \int_0^{\mathfrak{Z}}G_p(t)p_p(t)dt=\int_0^{\mathfrak{Z}}\Big[\sum_{r=1}^{\tau}\sum_{q=1}^{\tau}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt\Big]dt_0.$$

In any practical example, we prefer to think of the production unit as fixed, but we also like to divide the whole production process into stages that correspond to the essentially different processes on the production unit. It is possible to choose n[8a] and τ to satisfy these conditions, and there is no *a priori* relationship between n, τ, and T. For the purpose of simplification, a production unit might be defined as the production that passes through a stage in a unit of time. This definition, together with the assumption of uniform production, makes $n=\tau=$units of time in T. This seems objectionable, since a natural division into stages requires a variable production unit; on the other hand, if the production unit is fixed, the different stages are determined with no regard to the distinct processes of production. In order to avoid either a variable production unit or an artificial choice of stages, it seems best to regard the stages as defined according to convenience.

Before proving a fundamental theorem, the following lemma must be established.

LEMMA.

$$\int_0^{\mathfrak{Z}}G_p(t)p_p(t)dt=\int_0^{\mathfrak{Z}}\Big[\sum_{q=1}^{\tau}(\tau-q+1)\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt\Big]dt_0.$$

Proof: from (A),

$$\int_0^{\mathfrak{Z}}G_p(t)p_p(t)dt=\int_0^{\mathfrak{Z}}\Big[\sum_{r=1}^{\tau}\sum_{q=1}^{\tau}\int_{t_{q-1}}^{t_q}\{p_l^{(q)}(t_0+t)\dot{l}^{(q)}(t_0+t)+p_s^{(q)}(t_0+t)\dot{s}^{(q)}(t_0+t)\}\,dt\Big]dt_0,$$

[8a] The symbol n stands for the number of production units in the production process at time t.

$$= \int_0^{\mathfrak{J}} \left[\sum_{q=1}^{\tau} \sum_{r=q}^{\tau} \int_{t_{q-1}}^{t_q} \{ \quad \} dt \right] dt_0$$

$$\left[\text{Since } \sum_{k=1}^{n} \sum_{j=1}^{k} a_k a_j = \sum_{j=1}^{n} \sum_{k=j}^{n} a_k a_j \right],$$

$$= \int_0^{\mathfrak{J}} \left[\sum_{q=1}^{\tau} \int_{t_{q-1}}^{t_q} \{ \quad \} dt \sum_{r=q}^{\tau} (1) \right] dt_0,$$

$$= \int_0^{\mathfrak{J}} \left[\sum_{q=1}^{\tau} (\tau - q + 1) \int_{t_{q-1}}^{t_q} \{ \quad \} dt \right] dt_0.$$

The ability to express the total value of producers' goods produced during a period of time in this manner is necessary to all that follows, and it is made possible by the mathematical transformation of interchanging the order of summation.

We may now state and prove the basic theorem in Hayek's theory; we wish to emphasize the conditions necessary for its validity.

THEOREM.—*A transition to more (less) capitalistic means of production will take place if the total demand for producers' goods increases (decreases) relative to the total demand for consumers' goods.* More accurately, with our assumptions,

$$L^{\tau}(\mathfrak{J}) = \frac{\int_0^{\mathfrak{J}} D_p(t)dt}{\int_0^{\mathfrak{J}} D_c(t)dt}.$$

Proof:

$L^{\tau}(\mathfrak{J})$

$$= \frac{\int_0^{\mathfrak{J}} \left[\sum_{q=1}^{\tau} (\tau - q + 1) \int_{t_{q-1}}^{t_q} \{ p_l^{(q)}(t_0+t) \dot{l}^{(q)}(t_0+t) + p_s^{(q)}(t_0+t) \dot{s}^{(q)}(t_0+t) \} dt \right] dt_0}{\int_0^{\mathfrak{J}} \left[\sum_{q=1}^{\tau} \int_{t_{q-1}}^{t_q} \{ p_l^{(q)}(t_0+t) \dot{l}^{(q)}(t_0+t) + p_s^{(q)}(t_0+t) \dot{s}^{(q)}(t_0+t) \} dt \right] dt_0}$$

$$= \frac{\int_0^{\mathfrak{J}} G_p(t) p_p(t) dt}{\int_0^{\mathfrak{J}} I(t) dt}, \quad [\text{By lemma}] \quad = \frac{\int_0^{\mathfrak{J}} G_p(t) p_p(t) dt}{\int_0^{\mathfrak{J}} \{ D_c(t) + E(t) - F_c(t) \} dt}, \quad [\text{By (3.2)}]$$

$$= \frac{\int_0^{\mathfrak{J}} D_p(t) dt}{\int_0^{\mathfrak{J}} D_c(t) dt}. \qquad [\text{By (3.1) and (3.6)}]$$

The equality of the denominators in the first line and second line of the proof requires the same hypothesis of slow motion as (A). As stated in (2.9), this theorem is not true for the first definition of length of production; its validity also requires the special condition, $E(t) - F_c(t) \equiv 0$.

Incidentally, if $E(t)-F_c(t)$ is not identically zero, we have shown that

$$(4.1)\qquad L^r(\mathfrak{I}) = \frac{\int_0^{\mathfrak{I}} D_p(t)dt}{\int_0^{\mathfrak{I}} I(t)dt}.$$

If the interval of time $(0\mathfrak{I})$ is short with respect to the time for changes of $E(t)$, $F_c(t)$, and $I(t)$, we may write

$$(4.2)\qquad L^r(t) = \frac{D_p(t)}{I(t)},$$

which reduces in the case $E(t)-F_c(t)\equiv 0$ to the relation

$$(4.3)\qquad L^r(t) = \frac{D_p(t)}{D_c(t)}.$$

If preferable, of course, these quantities in numerator and denominator may be regarded as averages of their respective integrals.

5. *Comparison of the changes in $L^r(t)$ and the ratio* $\frac{D_p(t)}{D_c(t)}$.—Dr. Hayek considers the changes in $L^r(t)$ and changes in the ratio $\frac{D_p(t)}{D_c(t)}$ as equivalent; it is interesting to compare these changes. From (4.2), we have

$$(5.1)\qquad \begin{aligned} d\{L^r(t)\} &= \frac{I(t)dD_p(t) - D_p(t)dI(t)}{[I(t)]^2}, \\ d\left\{\frac{D_p(t)}{D_c(t)}\right\} &= \frac{D_c(t)dD_p(t) - D_p(t)dD_c(t)}{[D_c(t)]^2}. \end{aligned}$$

Recalling the hypotheses of no hoarding and neutral money, we have

$$\begin{aligned} D_c(t) &= I(t) - E(t) + F_c(t), \\ D_p(t) + D_c(t) &= M(t)V(t) + M'(t)V'(t) \equiv \text{constant}, \end{aligned}$$

where these yield

$$(5.2)\qquad \begin{aligned} dD_c(t) &= dI(t) - dE(t) + dF_c(t), \\ dD_p(t) &= -\, dI(t) + dE(t) - dF_c(t). \end{aligned}$$

For the sake of brevity, the discussion is limited to the simplest cases.

Case 1.—We hold $I(t)$ and $F_c(t)$ constant and allow $E(t)$ to vary. The differential equations (5.1) and (5.2) give in this case

$$d\{L^r(t)\} = \frac{dE(t)}{I(t)}, \qquad d\left\{\frac{D_p(t)}{D_c(t)}\right\} = \frac{[D_c(t) + D_p(t)]dE(t)}{[D_c(t)]^2}.$$

This shows that both $L^r(t)$ and $\frac{D_p(t)}{D_c(t)}$ vary in the same sense as $E(t)$, as Dr. Hayek states, but $|d\{L^r(t)\}| < \left|d\left\{\frac{D_p(t)}{D_c(t)}\right\}\right|$ for any $dE(t) \neq 0$ and $E(t) - F_c(t) \geqq 0$.

Case 2.—We hold $I(t)$ and $E(t)$ constant and allow $F_c(t)$ to vary. This situation gives

$$d\{L^r(t)\} = \frac{-\,dF_c(t)}{I(t)}, \qquad d\left\{\frac{D_p(t)}{D_c(t)}\right\} = \frac{-\,[D_c(t) + D_p(t)]dF_c(t)}{[D_c(t)]^2}.$$

This also agrees with Dr. Hayek's conclusion that the length of production is decreased as a result of increasing loans to consumers. It is clear that when $E(t)$ and $F_c(t)$ are both permitted to vary they may neutralize each other. As before, $|d\{L^r(t)\}| < \left|d\left\{\frac{D_p(t)}{D_c(t)}\right\}\right|$ for any $dF_c(t) \neq 0$ and $E(t) - F_c(t) \geqq 0$.

Case 3.—We hold $E(t)$ and $F_c(t)$ constant and allow $I(t)$ to vary. The differential equations (5.1) and (5.2) yield

$$d\{L^r(t)\} = \frac{-\,[I(t) + D_p(t)]dI(t)}{[I(t)]^2},$$

$$d\left\{\frac{D_p(t)}{D_c(t)}\right\} = \frac{-\,[D_c(t) + D_p(t)]dI(t)}{[D_c(t)]^2}.$$

Hence, both $L^r(t)$ and $\frac{D_p(t)}{D_c(t)}$ vary oppositely to $I(t)$ and

$$|d\{L^r(t)\}| \leqq \left|d\left\{\frac{D_p(t)}{D_c(t)}\right\}\right| \text{ as } E(t) - F_c(t) \leqq 0, \text{ for any } dI(t) \neq 0.$$

In his demonstration that an increase in bank credit leads eventually to a shortening in the length of production, Dr. Hayek considers that an increase in income has the same effect as an increase in credit to consumers.[9] A comparison of the last two cases shows that equal changes

[9] "Capital and Industrial Fluctuations," ECONOMETRICA II, 1934, 152–167. See p. 159.

in $F_c(t)$ and $I(t)$ have exactly the same effect on the ratio $\dfrac{D_p(t)}{D_c(t)}$, and the variations in $L^r(t)$ are in the same direction but they are not identical. We always have

$$\left|\frac{d\{L^r(t)\}}{dF_c(t)}\right| < \left|\frac{d\{L^r(t)\}}{dI(t)}\right|.$$

It is desirable to mention that $\dfrac{D_p(t)}{D_c(t)}$ is a good approximation to $L^r(t)$ if $E(t)-F_c(t)$ is small in absolute value. If this difference is zero, $L^r(t)=\dfrac{D_p(t)}{D_c(t)}$, but, as $E(t)-F_c(t)$ varies, the difference $L^r(t)-\dfrac{D_p(t)}{D_c(t)}$ varies in the opposite direction. Possibly Dr. Hayek has tacitly assumed $E(t)-F_c(t)=0$ in equilibrium; in fact, there does not seem to be any place in his diagrams in *Prices and Production* for the quantity $E(t)-F_c(t)$.

One may easily calculate the expressions for changes in $L^r(t)$ and $\dfrac{D_p(t)}{D_c(t)}$ while any two or all three symbols $I(t)$, $E(t)$, and $F_c(t)$, vary. It seems impossible to determine the result of changes in bank loans to producers without introducing an approximation that we wish to avoid at this time. So, with this brief discussion, we abandon the comparison of $L^r(t)$ and $\dfrac{D_p(t)}{D_c(t)}$, and proceed to a study of the stability of $L^r(t)$.

6. *Permanence of a change in $L^r(t)$.*—As before, we have the assumption of no hoarding and the equation of exchange to be satisfied at all times. That is,

$$\begin{aligned} D_c(t) &= I(t) - E(t) + F_c(t), \\ D_p(t) + D_c(t) &= M(t)V(t) + M'(t)V'(t) = m(t). \end{aligned} \tag{6.1}$$

One may make any assumption in addition to these that he deems appropriate for the practical situation, but for the sake of the logic of the situation he must not make such assumptions tacitly. In studying the behavior of $L^r(t)$ over a period of time, it is desirable to make an assumption regarding income. We neglect all the influences on income other than voluntary savings and payments on producers' goods. It is reasonable to suppose that income varies in some relation to past vol-

untary savings. There is also quite general agreement that income varies with past and present payments on producers' goods. The time lag depends on the stage in which the change in $D_p(t)$ occurs and the proportion of money payments in this and earlier stages that goes into income. These time lags, together with the lag in the effects of voluntary savings, are, in general, different; but, in order to simplify the discussion, we take them equal and assume that the variation in income is a linear combination of the variations in voluntary savings and payments on producers' goods after a time lag α. Thus, α is an "average" of the lengths of time it takes changes in various parts of $D_p(t)$ to reach $I(t)$.[10]

$$(6.2) \qquad dI(t) = kdE(t - \alpha) + adD_p(t - \alpha).$$

After carrying through the deductions, we may determine the consequences of different economic assumptions by giving the constants of proportionality special values in the general result. It is desirable to have some idea of a reasonable range for the constants. Certainly a must be limited to the interval (01), otherwise an increase in total money payments on producers' goods would decrease future income, or increase future income even more than the total increase in money payments on producers' goods. Surely the absolute value of k should be less than one, but there are arguments for considering k either positive, zero, or negative.

In order to study the changes in the length of production due to variations in voluntary savings and the effective supply of money, we assume equilibrium before a fixed time t_0; that is,

$$(6.3) \quad dE(t) = dI(t) = dD_p(t) = dD_c(t) = dm(t) = dF_c(t) = 0 \quad \text{for } t < t_0.$$

During the interval (t_0, t_1), voluntary savings and total money payments are allowed to vary by any given amounts ϵ and δ respectively, but they become constant again after time t_1. In order to study the most important case in practice, the quantity "net loans to consumers" is held zero at all times, so that the change in total money payments is entirely in the demand for producers' goods. In using the symbol $m(t)$ for total money payments, no distinction is made between variations due to any combination of changes in $M(t)$, $V(t)$, $M'(t)$, and $V'(t)$. It appears impossible to set up an equation in which the isolated

[10] There are reasons for using $dI(t) = \kappa dE(t-\alpha) + a[dD_p(t-\alpha) - dI(t-\alpha)]$ or $dI(t) = \kappa dE(t-\alpha) + a[dD_p(t-\alpha) - dI(t-\alpha)] + rdD_p(t)$; but the results are not essentially different and the formulas are more complicated.

variations in $F_p(t)$ may be studied without introducing more assumptions than we wish at this time.

We have

$$\text{(6.4)} \qquad \int_{t_0}^{t_1} dE(t) = \epsilon; \qquad \int_{t_0}^{t_1} dm(t) = \delta;$$

$$F_c(t) \equiv 0; \; dm(t) = dE(t) = 0 \text{ for } t > t_1.$$

To be definite, we consider $\alpha > t_1 - t_0$ in the detailed discussion. If $\alpha = t_1 - t_0$, the intervals $[t_1 + n\alpha, \; t_0 + (n+1)\alpha]$ vanish. If $\alpha < t_1 - t_0$, the ultimate results are the same as we obtain, but the variables cannot be definitely determined at the times $(t_0 + \alpha)$ and (t_1), because the integrals $\int_{t_0}^{t_0+\alpha} dE(t)$ and $\int_{t_0}^{t_0+\alpha} dm(t)$ are unknown.

It is necessary to separate the discussion into distinct periods.

(a) Interval $(t_0 \leqq t \leqq t_1)$. We have from (6.1), (6.2), (6.3), and (6.4),

$$dD_p(t) = dm(t) + dE(t); \; dI(t) = dF_c(t) = 0; \, dD_c(t) = -dE(t).$$

This gives

$$L^r(t) = \frac{D_p(t_0) + \int_{t_0}^{t} dm(t) + \int_{t_0}^{t} dE(t)}{I(t_0)},$$

and at the end of the interval

$$L^r(t_1) = \frac{D_p(t_0) + \delta + \epsilon}{I(t_0)} = L^r(t_0) + \frac{\delta + \epsilon}{I(t_0)}.$$

(b) Interval $(t_1 \leqq t \leqq t_0 + \alpha)$.

$$dD_p(t) = dI(t) = dD_c(t) = dE(t) = dm(t) = dF_c(t) = 0.$$

$$L^r(t) = L^r(t_1).$$

(c) Interval $(t_0 + \alpha) \leqq t \leqq t_1 + \alpha)$.

$$\begin{aligned} dI(t) &= k dE(t - \alpha) + a dD_p(t - \alpha) \\ &= k dE(t - \alpha) + a[dE(t - \alpha) + dm(t - \alpha)]. \\ dD_p(t) &= - dI(t). \end{aligned}$$

$$L^r(t_1 + \alpha) = \frac{D_p(t_0) + \delta + \epsilon - [k\epsilon + a(\epsilon + \delta)]}{I(t_0) + [k\epsilon + a(\epsilon + \delta)]}.$$

It is evident that $D_p(t)$, $I(t)$, and $L^r(t)$, are constant for all periods $[t_1 + (n-1)\alpha \leqq t \leqq t_0 + n\alpha]$, $n = 1, 2, 3, \cdots$. For the changes in $D_p(t)$ and $I(t)$ during the other periods, we have

$$\mathop{\Delta}_{t_0}^{t_1} D_p(t) = \delta + \epsilon; \quad \mathop{\Delta}_{t_0+n\alpha}^{t_1+n\alpha} D_p(t) = (-1)^n a^{n-1}[k\epsilon + a(\epsilon + \delta)].$$

$$n = 1, 2, \cdots$$

$$\mathop{\Delta}_{t_0}^{t_1} I(t) = 0; \quad \mathop{\Delta}_{t_0+n\alpha}^{t_1+n\alpha} I(t) = (-1)^{n-1} a^{n-1}[k\epsilon + a(\epsilon + \delta)].$$

Summing

$$I(t_1 + n\alpha) = I(t_0) + \sum_{i=0}^{n} \mathop{\Delta}_{t_0+i\alpha}^{t_1+i\alpha} I(t) = I(t_0) + \sum_{i=1}^{n} (-a)^{i-1}[k\epsilon + a(\epsilon + \delta)],$$

$$D_p(t_1 + n\alpha) = D_p(t_0) + \sum_{i=0}^{n} \mathop{\Delta}_{t_0+i\alpha}^{t_1+i\alpha} D_p(t) = D_p(t_0) + \delta + \epsilon$$

$$- \sum_{i=1}^{n} (-a)^{i-1}[k\epsilon + a(\epsilon + \delta)].$$

Letting t become infinite,

$$\lim_{t\to\infty} I(t) = \lim_{n\to\infty} I(t_1 + n\alpha) = I(t_0) + \frac{k\epsilon + a(\epsilon + \delta)}{1 + a},$$

$$\lim_{t\to\infty} D_p(t) = \lim_{n\to\infty} D_p(t_1 + n\alpha) = D_p(t_0) + \delta + \epsilon - \frac{k\epsilon + a(\epsilon + \delta)}{1 + a}$$

$$= D_p(t_0) + \frac{\delta + \epsilon - k\epsilon}{1 + a},$$

$$\lim_{t\to\infty} L^r(t) = \frac{D_p(t_0) + \dfrac{\delta + \epsilon - k\epsilon}{1 + a}}{I(t_0) + \dfrac{k\epsilon + a(\epsilon + \delta)}{1 + a}}.$$

In the general case, it is easier to follow a broken line graph of our variables than to discuss the analytical formulas. The values of the constants are all considered positive in this diagram, that is, the period (t_0t_1) is a period of expansion or boom. The exact values of the variables are known throughout the periods $[t_1+(n-1)\alpha \leqq t \leqq t_0+n\alpha]$, but they are known only at the end points of the intervals $[t_0+n\alpha \leqq t \leqq t_1+n\alpha]$. These values at the end points are connected by straight lines. If the integrals, $\int_{t_0}^{t} dm(t)$ and $\int_{t_0}^{t} dE(t)$, were known for all values of t in the interval (t_0t_1), then $D_p(t)$, $I(t)$, and $D_c(t)$, would have known values at all times.

Let us consider some of the more interesting economic hypotheses. First, we hold money neutral $(\delta = 0)$ and consider an increase in voluntary savings $(\epsilon > 0)$ during the interval (t_0t_1). Setting $k = a = 0$, we have

total income constant and an increase of $L^r(t)$ during the interval (t_0t_1) that is permanent.

However, if we wish to follow Hayek in this case, we set $a=0$ and $k<0$, because the additional money required for payments between en-

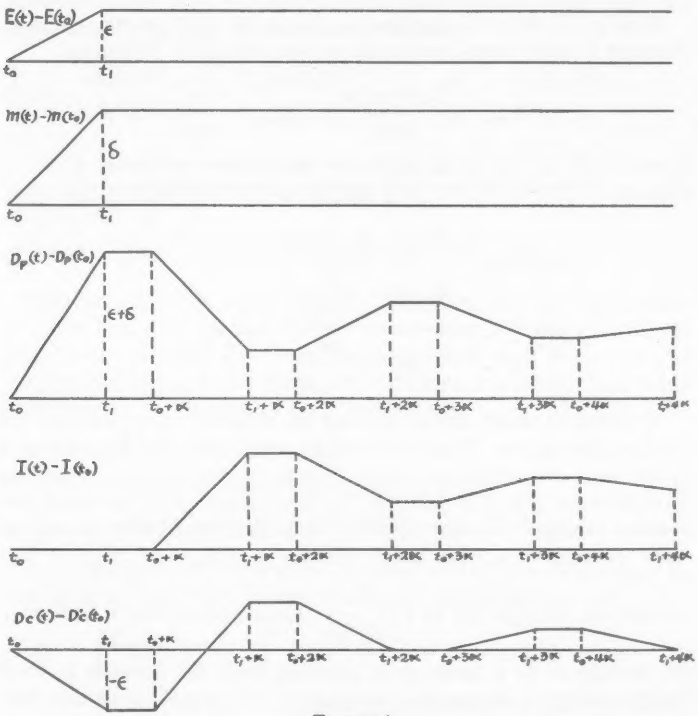


FIGURE 1

trepreneurs after the number of stages has increased is not available for income at a later date. Using these values for the constants, $L^r(t)$ is permanently increased, in agreement with Hayek's conclusion.

$$L^r(t_1) = L^r(t_0) + \frac{\epsilon}{I(t_0)} = L^r(t_0 + \alpha);$$

$$L^r(t_1 + \alpha) = \frac{D_p(t_0) + \epsilon - k\epsilon}{I(t_0) + k\epsilon} = L^r(t) \text{ for } t > t_1 + \alpha.$$

On the other hand, it may be argued that those individuals who vol-

untarily save will have increased money incomes at a later date, so that $k>0$. This gives

$$L^r(t) \gtreqless L^r(t_0) \text{ for } t > t_1 + \alpha, \text{ as } \frac{I(t_0)}{D_p(t_0) + I(t_0)} \gtreqless k.$$

Finally, in order to consider the change in $L^r(t)$ resulting from an increase in total money payments, we set $\epsilon=0$, $\delta>0$. This gives

$$L^r(t_1) = \frac{D_p(t_0) + \delta}{I(t_0)} = L^r(t_0) + \frac{\delta}{I(t_0)} = L^r(t_0 + \alpha),$$

$$\begin{aligned} L^r(t_1 + n\alpha) &= \frac{D_p(t_0) + \delta - a\delta + a^2\delta - \cdots (-a)^n\delta}{I(t_0) + a\delta - a^2\delta + \cdots (-1)^{n-1}(a)^n\delta} \\ &= L(t_0 + (n+1)\alpha), \end{aligned}$$

$$\lim_{t\to\infty} L^r(t) = \frac{D_p(t_0) + \dfrac{\delta}{1+a}}{I(t_0) + \dfrac{a\delta}{1+a}}.$$

A notion of these oscillations may be obtained by a glance at the broken line graphs. These conclusions agree with Dr. Hayek's in a rough way; at least the increase in $L^r(t)$ during the period of expansion (t_0t_1) does not persist unchanged. The comparison of the length of production at any time (when its value can be calculated) with its original value depends on a relationship between a, $D_p(t_0)$, and $I(t_0)$. In particular, $\lim_{t\to\infty} L^r(t) \gtreqless L^r(t_0)$ as $a \lesseqgtr \dfrac{I(t_0)}{D_p(t_0)}$. These fluctuations in $L^r(t)$ may be thought of as a trade cycle resulting from the increase in total money payments during the period (t_0t_1). We should recall that real income, for a fixed amount of labor and land, varies with $L^r(t)$ because of the assumption, (3.7), $\dfrac{\partial G_c(t)}{\partial L}>0$.

7. *An alternative definition of neutral money.*—We now introduce a new definition of neutral money and certain arbitrary but reasonable assumptions that yield Dr. Hayek's principal conclusions; and they enable us to study isolated variations in $F_p(t)$. Let us consider money neutral when $F_p(t) \equiv F_c(t) \equiv 0$. We also separate the money side of the equation of exchange into three parts; the first expression is the total money payments on producers' goods per unit time by means of credit,

the second is the total money payments on consumers' goods per unit time by means of credit, and the third is all other money payments.

$$(7.1) \qquad D_p(t) + D_c(t) = M_p(t)V_p(t) + M_c(t)V_c(t) + M(t)V(t).$$

Although there is no numerical relationship between these first two terms and $F_p(t)$, $F_c(t)$, it seems fair to assume

$$(7.2) \qquad \begin{aligned} M_p(t)dV_p(t) + V_p(t)dM_p(t) &\gtreqless 0 \qquad \text{as} \qquad F_p(t) \gtreqless 0, \\ M_c(t)dV_c(t) + V_c(t)dM_c(t) &\gtreqless 0 \qquad \text{as} \qquad F_c(t) \gtreqless 0. \end{aligned}$$

In this discussion we propose to neglect all changes in $D_p(t)$ except those due to voluntary savings and the first term in the second member of (7.1), that is,

$$(7.3) \qquad dD_p(t) = dE(t) + M_p(t)dV_p(t) + V_p(t)dM_p(t).$$

This section is concerned with Dr. Hayek's thesis that an increase in the length of production due to an increase in voluntary savings is permanent while that due to an increase in bank loans to producers is temporary (Hayek, p. 52). Hence, we consider the quantity, "net loans to consumers," as identically zero. We have from (3.2), (6.2), and (7.3),

$$(7.4) \qquad \begin{aligned} dD_c(t) &= dI(t) - dE(t), \\ dI(t) &= kdE(t - \alpha) + adD_p(t - \alpha), \\ dD_p(t) &= dE(t) + M_p(t)dV_p(t) + V_p(t)dM_p(t). \end{aligned}$$

As in the previous section, we consider equilibrium before time t_0, and allow $E(t)$ and $F_p(t)$ to vary during the period (t_0t_1); so that

$$(7.5) \qquad \int_{t_0}^{t_1} dE(t) = \epsilon; \quad \int_{t_0}^{t_1} M_p(t)dV_p(t) + V_p(t)dM_p(t) = \beta;$$

$$M_c(t)dV_c(t) + V_c(t)dM_c(t) \equiv 0 \text{ (from 7.2 and } F_c(t) \equiv 0).$$

After time t_1 we hold voluntary savings constant and money neutral in this new sense $(F_p(t) \equiv F_c(t) \equiv 0)$.

Equations (7.4) and (7.5) give the following results:

$$L^r(t_1) = \frac{D_p(t_0) + \epsilon + \beta}{I(t_0)} = L^r(t_0) + \frac{\epsilon + \beta}{I(t_0)} = L^r(t_0 + \alpha),$$

$$L^r(t) = L^r(t_1 + \alpha) = \frac{D_p(t_0) + \epsilon + \beta}{I(t_0) + k\epsilon + a[\epsilon + \beta]}, \text{ for } t > t_1 + \alpha.$$

If money is held neutral in the sense of the present discussion during

the interval (t_0t_1), $\beta=0$ and $L^r(t) \gtreqless L^r(t_0)$ for $t>t_1+\alpha$ as $\dfrac{I(t_0)}{D_p(t_0)} \gtreqless \kappa+a$,

if $\epsilon>0$. For this discussion Dr. Hayek considers $k<0$ and $a=0$, so that $L^r(t)$ is permanently increased. In order to obtain the results of an increase in bank loans to producers, we set $\epsilon=0$ and $\beta>0$. The length of production increases during the interval (t_0t_1); but, as stated by Dr. Hayek, it decreases throughout the period $(t_0+\alpha,\ t_1+\alpha)$. The final value of $L^r(t)$ as compared with the initial value depends on a relationship between a, $I(t_0)$, and $D_p(t_0)$; that is,

$$L^r(t) \gtreqless L^r(t_0) \quad \text{for} \quad t > t_1 + \alpha, \quad \text{as} \quad \frac{I(t_0)}{D_p(t_0)} \gtreqless a.$$

8. *Constant rate of increase of bank credit.*—In ECONOMETRICA,[11] Hansen and Tout state that the original lengthening of the structure of production due to an increase in bank credit to producers may be maintained by a steady increase in bank credit. Hayek interprets this to mean a constant rate of increase of bank credit $(F_p(t) \equiv \text{constant} > 0)$, and he believes the structure of production would ultimately shorten even in this case.[12] Let us investigate this with the equations (7.4) and the additional hypothesis,

$$\text{(8.1)} \quad \frac{d}{dt}\{M_p(t)V_p(t)\} \equiv \delta' \equiv \text{constant} > 0, \text{if } F_p(t) \equiv \text{constant} > 0.$$

In order to isolate the influence of $F_p(t)$, we set $dE(t) \equiv F_c(t) \equiv 0$ and we consider the system in equilibrium before time t_0. Equations (7.4) now become for all $t>t_0$,

$$\text{(8.2)} \quad \begin{aligned} dD_c(t) &= dI(t), \\ dI(t) &= adD_p(t-\alpha) \\ dD_p(t) &= \delta' dt. \end{aligned}$$

From (5.1) and the assumption of equilibrium before t_0, we have

$$dL^r(t) = \frac{\delta'}{I(t_0)}\,dt > 0 \text{ and } L^r(t_0+\alpha) = \frac{D_p(t_0) + \delta'\alpha}{I(t_0)} = L^r(t_0) + \frac{\delta'\alpha}{I(t_0)},$$

for t in the interval t_0 to $t_0+\alpha$.

[11] "Annual Survey of Business Cycle Theory," ECONOMETRICA, 1933, 119–147. See p. 140.

[12] "Capital and Industrial Fluctuations," ECONOMETRICA, 1934, 152–167. See p. 160.

After time $(t_0+\alpha)$,

$$L^r(t) = \frac{D_p(t_0+\alpha) + \delta' \int_{t_0+\alpha}^{t} dt}{I(t_0+\alpha) + \int_{t_0+\alpha}^{t} dI(t)},$$

$$= \frac{D_p(t_0+\alpha) + \delta'(t - t_0 - \alpha)}{I(t_0+\alpha) + a\delta'(t - t_0 - \alpha)},$$

since $$\int_{t_0+\alpha}^{t} dI(t) = a \int_{t_0}^{t-\alpha} dD_p(t) = a\delta'(t - \alpha - t_0).$$

Hence,

$$\lim_{t\to\infty} L^r(t) = \frac{1}{a}.$$

For the change in $L^r(t)$ we have

$$dL^r(t) = \frac{\delta' I(t) - a\delta' D_p(t)}{[I(t)]^2}\, dt \gtreqless 0, \quad \text{as} \quad \frac{1}{a} \gtreqless \frac{D_p(t)}{I(t)};$$

in particular, with reference to the statement of Hansen and Tout,

$$dL^r(t_0+\alpha) \gtreqless 0, \quad \text{as} \quad \frac{1}{a} \gtreqless \frac{D_p(t_0) + \delta'\alpha}{I(t_0)}.$$

Hence $L^r(t)$ may increase, remain constant, or decrease, after time $t_0+\alpha$; the choice is determined by this inequality between a, $D_p(t_0)$, δ', α and $I(t_0)$.

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THE CONCEPTION OF INVARIANTS IN DYNAMIC ECONOMICS

By Hans Bolza

The purpose of the present note is to state the mathematical and physical definition of *invariant* in such a form as will make this notion generally applicable to observations ordered in time, and in particular applicable to economic variables.

In mathematics we are taught to call an invariant a quantity which retains its numerical values in spite of transformations of coordinates. Take, for example, the length of a bar in a Euclidean space. If the coordinate system chosen is x, y, z, a Cartesian one, then the length of the bar in question will be

$$x^2 + y^2 + z^2 = l^2.$$

This value of l is invariant against all orthogonal transformations of determinant 1 in our space. This means that, choosing any other system of coordinates (ξ, η, ζ) which can be derived by an orthogonal transformation from the former coordinates, we shall find that the sum of the squares of the new coordinates will again have the same value, namely,

$$\xi^2 + \eta^2 + \zeta^2 = l^2.$$

Another well-known invariant in physics is the relation between space and time in the theory of Einstein, according to which we are transforming not only the coordinates of space but also the coordinates of time. Let us call the foursome of primary coordinates x, y, z, and t, and that of the second system to be found by transformation ξ, η, ζ, and τ. The expression for the invariant is then

$$x^2 + y^2 + z^2 - c^2 \cdot t^2 = \xi^2 + \eta^2 + \zeta^2 - c^2\tau^2 = \text{const.},$$

where c is a constant.

An entirely different type of invariant occurs when we have *two* coordinates depending on *one* parameter, z. In such a case, we have two equations of the form,

$$x = x(z),$$
$$y = y(z).$$

A transformation in the above sense is then no longer feasible; we can, indeed, now find only one new equation between x and y, by eliminating the parameter.

In the special case where the parameter z represents the time factor, we shall find a new category of invariants. Let us assume that we have two time functions, $x = f_1(t)$, and $y = f_2(t)$.

By eliminating the time coordinate from the two equations, we shall find a new expression for x and y, viz.,

$$\phi(x, y) = 0.$$

Whereas the two single variables x and y are, on the above assumption, functions of time, the expression found by *elimination* of it will be independent of t and we shall be entitled to call such an expression invariant in time.

Such considerations lead us to the conception of *causality* in natural science. If we have registered two independent facts in nature as functions of time, we shall always be able to find, by elimination of the time-coordinate, a new equation which will correlate the two variables through an equation which we may consider as the causal relation. It may seem surprising that "causal relations," which are believed to be the fundamental basis of natural science, should be capable of such a purely mathematical interpretation.

Some physical examples will illustrate this. The well-known experiment of Joule can be reproduced as follows: a body of given weight is falling and moving, by means of a string, a screw turning in a water-basin. This contrivance transforms the mechanical energy into heat. The experiment should always be made in such a manner that the weight is falling without acceleration, in order to avoid disturbing movements in the water.

What we are measuring is the linear increase of the mechanical work produced by the falling body, and the linear increase in temperature of the water and, in consequence, of the caloric energy of the water. We can easily write the two following equations:

(1) the mechanical work $= M = a \cdot t,$

(2) heat generated in water $= W = \alpha \cdot t,$

when a and α are constants. Combining the two equations, we get

$$M = \frac{a}{\alpha} \cdot W.$$

The elimination of the time-coordinate t from the above equations immediately gives us the relation between mechanical and caloric energy. The ratio a/α is *the caloric equivalent,* which is equal to 427 mkg/calorie.

Now, if we repeat the experiment of Joule with the modification

that the body is falling with a certain acceleration, we shall observe the increase of the temperature and, in consequence, of the caloric energy of the water as a non-linear function of time. We now obtain:

$$M = M_0 + at + bt^2, \tag{3}$$

$$W = W_0 + \alpha t + \beta t^2, \tag{4}$$

where a, b, α, and β, are constants. This gives:

$$\frac{M - M_0}{W - W_0} = \frac{a + bt}{\alpha + \beta t}.$$

Here the values M_0 and W_0 are the initial values of the mechanical or, respectively, of the caloric energy. It is evident that for the purpose of energetical argumentation only the difference between the energetical values is essential. It is seen that the ratio of these two values is a rational function of time. In the special case that the determinant of the coefficients a, α, b, and β, is zero, we shall obtain a *constant* invariant in time. New and accurately repeated experiments are necessary to determine whether the condition $\begin{vmatrix} ab \\ \alpha\beta \end{vmatrix} = 0$ is actually fulfilled or not. In this latter case, it will be possible to find a new invariant by eliminating the time factor t from the two equations for M and W.

Such invariants can be found between a great number of physical data. We do not propose to go into the question whether or not such invariants will have a different meaning in the case that the physical data in question are in a perceptible degree connected in space, as they are in Joule's experiment, as compared with cases where the physical data are not correlated in such a way that we can see any so-called material or concrete causal connection between them.

An invariant expression seems to represent in any case a "causal" relation under the following conditions: if we find by observation a physical datum in its evolution throughout time, we can find the respective derived function by differentiating the observed variable with respect to time. Between the observed time-function and its calculated derivative we can eliminate the time coordinate and get a so-called "causal" relation between a time-variable and its derivative which can be proved by physical experiment. In this case we undoubtedly deal with the same fact under two different aspects, viz., in its integral and in its differential aspect. Let there be given the integral function

$$M = f_1(t); \tag{5}$$

then we find its derived function as

$$\frac{dM}{dt} \equiv L = f_2(t). \tag{6}$$

From these two equations, we can again derive an expression invariant in time by eliminating t, so that we obtain

$$\phi(M, L) = 0.$$

May I say, by the way, that we have to divide all our observations of the outer world into those two categories of integral and differential functions which I attempted to define at the fourth European meeting of the Econometric Society at Stresa. I have called these two categories "quantity function" and "intensity function" respectively.

As a first example of such an invariant in time between a "quantity function" and the respective "intensity function," let us consider what happens when a body is falling under the influence of gravitation. The quantity function expressing the distance covered, s, is then

$$s = s_0 + v_0 t + \frac{b}{2} t^2, \tag{7}$$

s_0 being the initial position of the body, v_0 the initial velocity, and b the well-known acceleration of Galileo's law having the numerical value of $b = 9.81$ m/sec.[2] The corresponding intensity function which, in this case, is identical with the *velocity* of the falling body will be

$$\frac{ds}{dt} = v = v_0 + bt. \tag{8}$$

By eliminating the time factor t from the two equations (7) and (8), we find the following invariant in time,

$$s \cdot b - \frac{v^2}{2} = s_0 b - \frac{v_0^2}{2} = \text{constant}. \tag{9}$$

Assuming a constant mass (m) of the body, we can multiply the equation by this value m without disturbing the invariant character of the equation. We obtain, with a slight modification, the expression

$$\frac{mv^2}{2} - ms \cdot b = C. \tag{10}$$

This is the well-known law of conservation of energy in the case of gravitation. The first term is the expression for the *kinetic energy*, and the negative value of the second term the expression for the *potential energy* of gravitation. It follows that we are entitled to interpret, in this example, the principle of conservation of energy as the expression,

invariant in time, between the distance (quantity function) and the velocity (intensity function) which is obtained by time-differentiating the former function, if we eliminate time from the two equations thus obtained. Before we proceed to a further generalization of this idea, we propose to offer a numerical example and to demonstrate graphically the meaning of such invariants in time.

Let the initial value of the position be $s_0 = 1m$, and that of the velocity $v_0 = 5$ m/sec. We shall then have the following expression for

the quantity function, $$s = 1 + 5\cdot t + \frac{9.81}{2}\cdot t^2,$$

the intensity function being $v = 5 + 9.81\cdot t$.

The meaning of the invariant character of the equation stating the conservation of energy is this: we can select at any moment, let us say after five seconds, the value of the intensity function (velocity) and the corresponding value of the quantity function (way). If we combine these two values according to the law stated by the principle of the conservation of energy, we shall always obtain the same, constant result, in our case 2.69 gr m²/sec.[2] This result presupposes that the mass in question has the value 1.

It is essential that this value has been found under the conditions determined by a certain initial state. If we select another initial state we may find for the invariant another numerical value, up to the limit zero.

When we deduced the factor invariant in time in our last example, we needed but *one* physical datum, either that of the quantity function or that of the intensity function. Owing to the correlation existing between these two functions, the second one can always be found by differentiation or by integration if the first function has been determined by observation. In the case of gravitation we find, for instance, that the intensity function (the velocity of the falling body) is a linear function of time. It is evident that, in all other cases where we observe the intensity function as a linear time function, we shall find the same invariant expression. Let us assume, for instance, that a light-bulb factory is producing and delivering to the trade an increasing output of bulbs. Let us again call the coefficient of acceleration of its production b. We shall then find that half the square of the intensity function of production, that is to say, the production per day or month, minus the total number of the delivered lamps, multiplied by the factor b, will be a constant. A numerical example will prove this assertion. The lamp factory may already have furnished 90,000 bulbs at the beginning of our observations. The intensity of production may at this moment be

1000 lamps per day, and the acceleration of the production may have been five lamps per day^2. Under these conditions, we shall find the total number of incandescent lamps furnished to have been $s = 90{,}000 + 1000 \cdot t + 5/2 \cdot t^2$ (this being the quantity function), and the intensity of production will be found to have been v = output of lamps per day $= 1000 + 5 \cdot t$ (this being the intensity function). The factor invariant in time is determined by the formula:

$$\frac{v^2}{2} - s \cdot 5 = \text{const.} = \frac{v_0^2}{2} - s_0 \cdot 5.$$

We shall easily find the numerical value of this expression to be 50,000 (lamps per day)^2. If we make the same calculation at any later date, using the total number of lamps delivered up to that date and taking into account the respective intensity of production, we shall always obtain the same value.

In this example, it is even more evident that this invariant must have a positive value, since it contains the square of the material product called "lamps." But there is a slight difference between this case and the above analysed example of gravitation. For the straight line which the falling body is running through is—at least theoretically—unlimited in one direction, though it may be limited in the other direction. But any material body is limited in its temporal existence and, therefore, the stock of any material thing has limits in both directions. Every stock can always be represented as the difference between two quantity functions, one quantity function indicating the total amount of units produced and the second quantity function indicating the total amount of units destroyed.

Professor Divisia has introduced the term "ensembles renouvelés" for such pairs of quantity functions. Irving Fisher has been using the expression "stock" for the difference between two quantity functions and the expression "flow" for the intensity function. In order, however, to preserve the terminological continuity of my various publications on this subject, I prefer to use henceforth the expression "intensity function" instead of "flow." The term "stock" will be applied to the difference between the two quantity functions.

We propose to demonstrate by a physical example the notion of "stock." Let us take the movement of a pendulum. What we really observe when we look at the movement of a pendulum are two quantity functions, one quantity function indicating the accumulated amount of distance the pendulum weight has passed in the one direction and the second quantity function indicating the respective value for the other direction of the swing. It must again be emphasized that such a

thing as a negative line or a negative distance traversed by the pendulum in movement does not exist in reality. The upper part of Figure 1 shows these quantity functions. When the pendulum swings from the left to the right side the quantity function Q_2 does not increase since, by definition, this latter quantity function indicates the total amount of distance of the pendulum weight swinging from the right

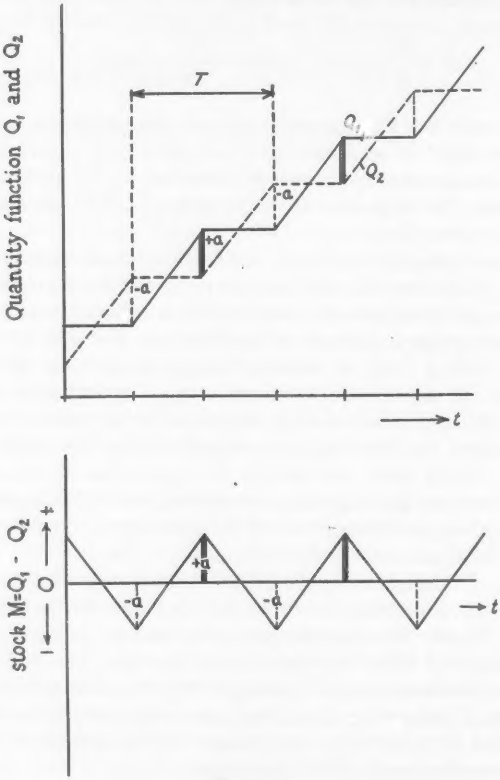

FIGURE 1

side to the left, and *vice versa*. The difference between these two quantity functions or, to use the expression of Irving Fisher, the "stock," is alternatively changing from a positive to a negative value if we have correctly selected the initial conditions. The periodical movements of this "stock" can be approximately described by means of a cosine function, so that we may write

$$M = a \cdot \cos kt; \; k = \frac{2\pi}{T}. \tag{11}$$

In this formula, a indicates the amplitude and T the period of the harmonical movement. In the same manner as before, we may try to find a correlation invariant in time between the "stock" and the derived function of this "stock." The latter will be

$$\frac{dM}{dt} = L = -a \cdot k \sin kt. \tag{12}$$

To eliminate the time coordinate t, we shall use the square of both equations so as to obtain the advantage of the formula

$$\sin^2 x + \cos^2 x = 1.$$

This gives

$$\begin{aligned} \frac{M^2 \cdot k^2}{2} &= \frac{a^2 \cdot k^2}{2} \cdot \cos^2 \cdot kt, \\ \frac{L^2}{2} &= \frac{a^2 \cdot k^2}{2} \cdot \sin^2 kt, \end{aligned} \tag{13}$$

and, hence,

$$\frac{L^2}{2} + \frac{M^2 \cdot k^2}{2} = \frac{a^2 \cdot k^2}{2} = \text{constant}. \tag{14}$$

This equation represents the characteristic "invariant in time" for the movement of a pendulum and is nothing else but the expression of the principle of the conservation of energy for a harmonical movement if the mass of the pendulum has the value 1. The first term is the expression for the kinetic energy, the second term the expression for the potential energy. Physicists know how to find the force of elasticity by derivating the potential energy with respect to the argument M, and how to find the equation of Newton by derivating the invariant equation with respect to time t. The condition for our being able to write down the Newtonian equation of movement is that a characteristic invariant in time does exist.

The conception of the invariant characteristics of a harmonic vibration can be adapted to the fluctuations of the means of payment, which have to be looked upon as a "stock."[1] We shall then have to

[1] In a recent book, *Ein neuer Weg zur Erforschung und Darstellung volkswirtschaftlicher Vorgänge*, Berlin: Julius Springer, 1935, I have tried to give a scientific definition for this conception.

find a relation invariant in time between the "stock" of circulating media and its derivative. According to the mean-value-theorem, there exists then an intermediate moment at which the value of the derivative of the "stock" is equal to the total amount of new credits created in one year, in one month, or in any other unit of time, or—conversely—the total amount of credits withdrawn in the same unit of time. To take the simplest case of a cyclical movement, let the "stock" (M) of circulating media as a function of time be approximately described by the expression:

$$M = M_0 + b \cdot \cos kt. \tag{15}$$

In this formula, M_0 means the average volume of means of payment around which the actual volume is fluctuating. The derivative of this is

$$\frac{dM}{dt} = L = -b \cdot k \cdot \sin kt. \tag{16}$$

The relation invariant in time between these two equations is

$$\frac{L^2}{2} + \frac{(M - M_0)^2 \cdot k^2}{2} = \frac{b^2 k^2}{2} = \text{const.} \tag{17}$$

If we believe that business improvement and depression are always proportional to the total amount of money of all kind in circulation (M) in a given economic area, we can conclude from (17) that the so-called trade-cycle is determined by the period in which credit creation and the withdrawal of credit is changing in the course of time. The intensity of a boom or a depression will then be determined by the amplitude of this periodical movement.

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NOTE ON THE TERM "ECONOMETRICS"

PROFESSOR TOMASZ LULEK of the University of Cracow, Poland, has communicated to the editor the following quotation from the book *Grundriss einer Oekonometrie und die auf der Nationaloekonomie aufgebaute natürliche Theorie der Buchhaltung*, by Pawel Ciompa, published in Lwow (Lernberg) in 1910:

"Wie die mechanischen, akustischen, dynamischen und dgl. Erscheinungen durch die Physik oder wie die Massenerscheinungen durch die Geometrie, so sollten auch die volkswirtschaftlichen Erscheinungen durch eine Lehre, die ich mir als eine Art Oekonomographie vorstelle, zur Darstellung gebracht werden. Diese Oekonomographie waere eine Art darstellende Nationaloekonomie, sie muesste auf der Nationaloekonomie, Mathematik und Goemetrie aufgebaut werden. Einer solchen Lehre wuerde dann vor allem die geometrische Darstellung des Wertes zufallen. Diesen Teil der Oekonomographie nenne ich Oekonometrie. Die praktische Anwendung dieser Oekonometrie auf die mathematische Darstellung der Werte und deren Veraenderungen ist dann die Buchhaltung. Umgekehrt ist dann die Oekonometrie nur die Theorie der Buchhaltung."

The quotation is interesting as evidence of an early use of the word "Oekonometrie." I used to think that this word, or more precisely, the corresponding French word "économétrie," had first been used in my 1926 paper, "Sur un problème d'économie pure."[1] The above quotation shows that this is not correct. It still seems, however, that, taken in the now accepted meaning, namely, as the unification of economic theory, statistics, and mathematics, the word was first employed in the 1926 paper. Pawel Ciompa seems to emphasise too much the descriptive side of what is now called econometrics.

RAGNAR FRISCH

[1] *Norsk Matematisk Forenings skrifter*, 1926.

OBITUARY

ECONOMETRICA records with deep regret that the deaths of the following members of the Econometric Society have occurred during the past year.

Herr Dr. Sigismund Gargas
The Hague
Holland

Professor Raymond Garver
University of California at Los Angeles
Los Angeles, California

Dr. Edv. Mackeprang
Köbmagergade
Copenhagen, Denmark

Professor Vincent Porri
30 Corso Peschiera
Turin 110, Italy

Professor François Simiand
27, Boulevard de la Tour-Mauborg
Paris, France

Professor Senjiro Takagi
Keiogijuka University
Tokyo, Japan

ANNUAL SURVEY OF ECONOMIC THEORY: THE SETTING OF THE CENTRAL PROBLEM[1]

By JOHAN ÅKERMAN

1. POSTULATES

THE SCIENCE of economics is essentially dualistic: value and matter, psychology and technic, price and quantity, are the perpetually recurring coordinates in the setting of economic problems. This all-embracing framework may be termed the *valuation-quantity relation*.

Next comes the question of how the different variables in the psychical and physical fields should be treated: as individuals, groups ("trading-bodies"), classes or nations; as specified goods, prices or price-levels, wages or wage-levels, interest-rates or levels of interest-rates. This constitutes the *classification and averaging problem*.

Any discussion of economic questions involves reference to some kind of community and certain assumptions in regard to the existing political, social and technical structure, these three factors being closely correlated and requiring, therefore, to be dealt with simultaneously. This constitutes our *institutional and structural premiss*.

It is further necessary to set forth clearly the fundamental instincts or motives which form the springs of action of individual groups and nations; such an exposition also involves, directly or indirectly, an indication of the trend of philosophical and ethical thought in our line of argument. We must, therefore, determine our *psychological premiss*.

In the discussion of the relations between different economic factors we must clearly determine what other factors, not included in the investigation as variables, should be regarded as of a subsidiary character and accordingly covered by a *ceteris paribus* postulate. We must, therefore, pose the *problem of constants*.

Finally, it should be clearly stated whether our reasoning falls under static economics, according to which the time element is ignored ("full mobility but no motion," i.e., no motion along the time-scale), or the domain of dynamic analysis, where the rates of increase of different variables are studied along with periodicities, time-lags and the introduction of new disturbing elements. In short, the *time-postulate* should be clearly formulated.

[1] The present survey has been written at the request of the editor as a survey on general economic theory. Since Dr. Åkerman has laid special emphasis on the time element and the equilibrium and disequilibrium problems, it has been found appropriate to let this paper replace in the survey program of ECONOMETRICA one of the surveys on business cycle theory. EDITOR.

When these six premisses have been established and explained, the problem of methodology is to a large extent determined, question, concepts and method being closely related. A survey of contributions to our science should, therefore, not primarily seek for "results," but for new ways of enquiring what the authors are asking, how they are asking and, in some measure, also why they are asking in such or such manner.

When the classical economists tried to ascertain the "cause" of value, and when the pioneers of the mathematical school formulated the logic of economic exchange, they were constantly endeavouring to give a comprehensive explanation of valuation and distribution. The element of choice, in the consumer's mind, had to be brought into connexion with the laws of the market, "demand" must balance "supply," also with the laws of competitive production, and hence with the distribution of wealth among the factors of production. There are indeed gaps between the different theories owing to differences in conception and the significance of fundamental concepts, but there has been an astonishing conservatism in the formulation of the central economic problem from Ricardo to Pareto. This conservatism shows that the original formulation was really leading up to a crucial theme: the individual and social valuation of scarce goods, and the consequences ensuing from this ever changing valuation in the enumeration of the factors of production. As a background to our discussion of new contributions the following table may be useful:

THEORIES OF ECONOMIC EQUILIBRIUM
PROCESS OF VALUATION AND DISTRIBUTION

	1. ⟵ Labour Cost ⟵ (Ricardo)	
	2. ⟵ Producer's Disutility ⟵ or Opportunity Cost (Marshall, Davenport)	Labour ⇅
Consumer Experiments to realise marginal utility level. "Economic man" (=labourers, capitalists, landowners).	3. ⟶ Utility ⇄ Cost ⟵ (Jevons)	Producer Experiments to maximise profit down to marginal productivity level. "Representative firm."
	4. ⇄ Utility ⇄ Cost ⇄ (Walras)	
	5. ⇄ Price = Cost ⇄ (Cassel)	⇅
	6. ⇄ Choice ⇄ (Pareto)	Capital and Land.

In this table the communication between consumer and producer marks the three fundamental questions of valuation and distribution which are still being hotly debated, viz., (1) What is the starting point of the deductive process, where should reasoning set out? (2) How can individual valuation be generalised to group-valuation, what is the connexion between marginal utility and social value? (3) What "tableau économique" gives the best picture of the relation between the general pricing process "on the market" and the production function, based on marginal productivity?

To a great extent the answer to these questions is anticipated by the formulation of the problem, and this formulation is already involved in the choice of the concepts which mark the development. The methodological problem is, in a great measure, the real problem. The greatest difficulty is perhaps involved in the fact that economic methodology is neither entirely an arbitrary exogenous question, as in history, nor an endogenous problem inherently correlated with the real problem, as in mathematics. It is a noteworthy and significant fact that the founders of the historical and mathematical schools, Schmoller and Walras, both laid stress on the double tracks in economic thinking, on inductive and deductive analysis.

If we are to understand the main stream of thought in recent economic theory, we must start from some kind of determination of our postulates, especially in view of the absence of such spadework in the greater part of pure economics. All our six premisses have been hotly debated in the last decade, but different tendencies in the reasoning may be discerned. The valuation-quantity relation has been brilliantly discussed by the adherents of the schools of Vienna and Lausanne, and illumined by their attempts to connect the marginal productivity concept with the simultaneous equations of the general pricing mechanism. The classification and averaging problem has made distinct progress by a deductive as well as inductive analysis of the meaning of index numbers; moreover, econometric investigations have thrown much light on the varying usefulness of traditional classification. The institutional and structural premisses have been kept in view by the various sociological authors but there has been very little connexion between this form of research and pure economics, in which these phenomena have been disregarded as being constants in the momentary production and consumption function. The debate which has been waged round the psychological premiss has consisted in a continued critique of the theory of subjective value and in constructive attempts to build a theory of choice on an inductive foundation. The problem of constants has too often been neglected in quantitative studies, which frequently start without these indispensable preliminaries.

But the most striking change in the formulation of the central economic problem is that which has taken place in our view of the time-element as a result of an increasing contact between business-cycle theory and general economic theory. Only twenty years ago the great body of economists would have asked: Is business-cycle theory conceivable? At present a more pertinent question seems to be: Is pure equilibrium theory conceivable, in the meaning of a workable general explanation of economic life?

Our discussion of the line of development in the formulation of the central economic problem will follow this part of the stream, where the current is most rapid, and bring the discussion of the other five premisses into connexion with this evolution of general theory. Dynamic problems of value, classification and averaging in a changing community, institutional transformation, psychological shift and conversion of the constant factors will thus be welded to the modifications of the equilibrium concept. Starting from partial alterations of the equilibrium theory we shall proceed to general formal modifications and terminate our study with a scrutiny of real modifications, which in some measure tend to convert the theory of equilibrium into a theory of time. We are thus ranging the various contributions according to falling dignities of neo-classicism.

2. THEORY OF DISTRIBUTION

The last ten years have witnessed a peculiar revival of those economic thoughts, conceived in the early seventies, which in many respects (political, social and monetary), had much in common with recent post war years. Walras' mathematical exposition of general economic equilibrium has been made accessible to all students of economics by Cassel's theory of the pricing process, in which, however, all psychological premisses have been truncated and superseded by the "principle of scarcity." Jevon's fundamental ideas on marginal utility and Menger's differentiations of present and future goods gained currency at the same time by a general study of Wicksell's investigation on "value, capital and interest," which had taken thirty years to win full recognition. In the nineteen-twenties an animated discussion thus arose between two schools of which both had their origin in 1871. This discussion, however, is much confused by the fact that the original common utility concept has been discarded in Cassel's neo-classic exposition, and that the Jevonian rather realistic time-element has been replaced by Böhm-Bawerk-Wicksell's "imaginary" time-concept, associated with the lengthening of the production process, but not with an empirical time-scale. Cassel's factor of production "capital-disposal" is, moreover, a purely time-free market-concept, even more so than

Walras' *service d'approvisionnement*. The time-premiss and the equilibrium concept being identical in the two schools, we thus have a controversy which in regard to the statement of the problem, though not with respect to questions such as the theory of capital where there is still great divergence of opinion, mainly centres round two points, the valuation process and the conception of marginal productivity.

The *subjective theory of value* has indeed been much criticised in recent years. The well known investigations of Max Weber,[2] Cornélissen,[3] and Benjamin Anderson,[4] have been followed up by fresh examination of the "circular reasoning" and the "economic-man" theory of hedonistic economics. Among several works of this nature the most stimulating and vigorous is undoubtedly an essay by G. Myrdal.[5] He attacks the psychological premiss more sharply than his predecessors. We have, he observes, no primary empirical material whereby to ascertain the real, deep-lying motives of individuals and any kind of "demand function" is in the ultimate merely a rationalisation a posteriori. Any kind of economics of welfare involving the "maximising" of the social product as an aim, therefore, lacks foundation, the real determinants in the valuation-free price equation being the "factors of social power." It is, moreover, impossible to average the valuations of individuals, and we have no means of distinguishing between rational and irrational behaviour, i.e., between rational and irrational valuation. As a central concept of economic science value should, therefore, be superseded by price.[6]

This kind of criticism starts and terminates in the domain of pure logical concepts. What it says is, in substance, this: How can empirical observations be connected at all with deductive concepts? This difficulty is not peculiar to the theory of marginal utility. Any rationalisation of observation of the outer world which assumes the form of a "law" must pass over this stumbling block. We can never study anything but manifestations, never any form of "thing-in-itself." A "law" is founded on the observation of regular recurrence, and becomes more firmly established according as this recurrence is regular and

[2] Max Weber, "Grenznutzentheorie und Psychophysik," *Archiv für Sozialwissenschaft*, Band 27, 1908, and *Gesammelte Aufsätze zur Wissenschaftslehre*, Tübingen, 1922.

[3] Ch. Cornélissen, *Théorie de la Valeur*, Paris, 1903.

[4] Benjamin Anderson, *Social Value*, Boston, 1913.

[5] Gunnar Myrdal, *Das politische Element in der national ökonomischen Doktrinbildung*, Berlin, 1932. Original Swedish edition, 1930.

[6] In a lucid study O. Morgenstern, *Schriften des Vereins für Sozialpolitik*, Band 183/I, 1931, p. 4, says, "In the theory of value everything has only one purpose and that is to help explain prices." The author thus underlines the *instrumental* character of the theory of value.

frequent. The theory of marginal utility is a statistical law of this nature.[7]

Against marginal utility the adherents of the neo-classic "value-free" school oppose "the fundamental fact of scarcity." But how has this "fact" been ascertained? It centers on a single point, the general equilibrium of the price mechanism, which at once rests on and expresses the principle of scarcity. It seems inconceivable that this "principle" could be detached from its position in the general price equilibrium. We cannot extend this principle, because "demand," "supply," "price" and "scarcity" are just integral concepts to be used in the formal logic of algebra, and the psychological premiss as well as time-element premiss are simply relegated to the limbo of constants. It is no use to go further back on the roads leading from price to demand, from demand to valuation, from valuation to behaviour, because there is no connexion between any of the facts revealed by observation and the concepts. Moreover the actual pricing is determined at the moment by the rules of general equilibrium, and any attempt to "explain" what is going on in some part of the mechanism is inconceivable, as the data used in this explanation are *ex ante* or *ex post*, and do not refer to the pricing at the given moment and in given circumstances. This all-embracing system cannot rest on the support of any process of deductive or inductive reasoning. We may think of the valuation process in discussing price equations, but we must not use the word value, which inevitably opens up new chains of causation, or, more strictly, new sequences of repetition phenomena.

We shall proceed to the second point at issue in the controversy between the neo-Walrasian and the neo-Jevonian schools, *the problem of marginal productivity* and its relation to the valuation and pricing process. One of those who first reverted to this fundamental problem in economic theory and has shown most acumen in its handling is W. L. Valk.[8] His treatise and a subsequent discussion with J. Schumpeter seem to serve as a starting point for recent developments in this sphere. Schumpeter says,[9] we may state the problem of valuation or pricing of the means of production in two ways, (1) as the production factors do not cover direct consumption, the valuation of these services must proceed on indirect lines, this leads to the theory of imputation, and particularly to that of v. Wieser, and (2) as the factors of production do not enter into the market of consumption goods, a measure of value in use

[7] Compare, however, F. Kaufmann, "On the Subject-Matter and Method of Economic Science," *Economica*, Nov. 1933.

[8] W. L. Valk, *The Principle of Wages*, London, 1928.

[9] J. Schumpeter, "Zur Frage der Grenzproduktivität," *Schmollers Jahrbuch*, 1927.

is inconceivable, and we must study the pricing of these services on their own market, where entrepreneurs will pay as little as possible, and the owners of the means of production will demand as much as possible. This question leads to the theory of general price equilibrium. Valk has set out to establish a synthesis between these two lines, but especially between the theories of J. B. Clark and Walras. The theory of marginal productivity is the theory of marginal utility applied to means of production; it is also a derivative valuation of consumption goods. At the same time, however, the actual bidding takes place on other markets, where the rules of competitive pricing prevail. Can these twofold aspects be converted into a single picture?

The theory of *marginal productivity* enters the workshop and studies the various combinations of production factors and the consequences which they involve in regard to the general pricing of services and goods. The theory of *general price equilibrium* sets out from the pricing of consumption goods and endeavours to generalize this process in a system which includes the bidding for means of production. The current of values which, in a theoretically infinite space of time, adjusts the tension between the various markets for means of production and the various markets for the products, sets out from the former, markets in the theory of marginal productivity, from the latter, markets in the theory of general price equilibrium. The productivity theory is an essentially *dynamic* concept, being associated with the rate of increase of capitalization, but is used in combination with an "imaginary time-factor," not a real time-scale. The theory of price equilibrium is essentially a *static* concept, being associated with the exchange of commodities and represented by price-tables without a time element of any kind in its system of coordinates.

The productivity theory is an *open* system; it starts from the sources of production factors at the periphery, *which may be expanded*, and converges on the prices of consumption goods; the repercussion of imputed product value on the means of production being the equalizing counter-current or reflux. The equilibrium theory is a *closed* system of simultaneous equations, in which values radiate to a stable periphery.

The most hotly debated part of Walras' theory, that part which he himself rejected in the final edition of *Éléments d'économie politique pure*, has given rise to fresh discussion. The theorem runs: "The whole quantity of the manufactured product is distributed among the productive services." Since Wicksteed's study on the coordination of the laws of distribution, published forty years ago, it has been assumed that this identity between product and means of production values will always be obtained when large and small scale production give the same return, that is, when the product of a proportional com-

bination of the means of production is increasing in a straight line. Zeuthen[10] and Schneider[11] have demonstrated that this is sufficient, but not necessary; there are other functions which will result in a distribution without residue. The Walrasian formulation of the theorem of marginal productivity, in its final form, may encourage a radical attempt to clear out the valuation side in the valuation-quantity relation. The starting point is the proposition: free competition brings the costs of production to a minimum (cost = price, at the margin). In that case the rate of remuneration of each factor of production is equal or proportional to its marginal productivity. If "cost" is regarded as a concept belonging to the economics of enterprise, and the classical, single-tracked, objective principle of cost and price accounting is taken on trust, then all the variables in the production function are objective, with the exception of the first. But the free competition premiss is treated as an a priori postulate, in fact as a constant. The marginal productivity concept leaves the door open for a treatment of the psychological premiss as well as for the study of the time-element, whilst the theory of general price equilibrium shuts the door on both. How far the door can be opened again will be discussed in the last section of this paper.

The theories of marginal productivity and price equilibrium have both the distinct merit of being true in any form of community. But they observe the production-consumption process from different angles. The former proceeds from the general to the partial, the latter the opposite way. It is doubtful whether there is much to gain by an attempt to change this stereoscopic picture into a one-eyed synthesis.[12]

3. THEORY OF MONEY

The intimate relation between general theory and monetary theory has never been more apparent in modern times than in the last few years, when a partial modification of equilibrium theory has been attempted by means of a restatement of monetary theory. Monetary theory and business cycle theory should, according to a prevalent view,

[10] F. Zeuthen, "Knappheitsprinzip, technische Kombination und ökonomische Qualität," *Zeitschrift für Nationalökonomie*, Band IV, p. 1.

[11] E. Schneider, "Bemerkungen zur Grenzproduktivitätstheorie," *Zeitschrift für Nationalökonomie*, Band IV, p. 604.

[12] In a review of Valk's book, K. H. Stephans declares that the author's attempt to weld Clark's marginal productivity theory to the Walras-Cassel theory of general equilibrium should be superseded by a combination of Clark's and Marshall's fundamental ideas; the marginal productivity theory should be completed by an analysis of demand and supply functions. This, however, would introduce the concept of *partial* equilibrium, which would scarcely answer the request for an "inclusive" theory. *Zeitschrift für Nationalökonomie*, Band V, p. 371.

at the same time form a line of union and a line of demarcation between static and dynamic method.

Before reviewing the most significant contributions of this kind we may state that the psychological, institutional and constants' premisses are in all monetary studies treated in almost the same manner. The psychological premiss consists of the concept of the "economic producer," the "economic banker," and the "economic consumer," each attempting to maximise profits and utility, the activities of central banks, however, being as a rule treated as a philanthropic business in pursuit of common welfare. The policy of commercial banks is rarely treated as an autonomous factor in the relation between savers and enterpreneurs; the investing process as such is regarded as a middleman's trade in the money and capital market.[13] The institutional premiss tends towards the general frame of free competition, and it is noteworthy that the theory of money has rarely been brought into contact with the theory of monopoly.[14] The constants' premiss has a peculiar meaning in the theory of money as there are pseudo-constants behind each variable in the quantity equation, and these are more easily brought into the orbit of discussion than the different *ceteris paribus* factors of general economic equilibrium, which are much more numerous and much more closely interrelated.[15]

A study of the setting of the problem in the outstanding treatises on money suggests that two incongruous elements are combined in all of them. The one is the idea of a normal state, inherited from the traditional thought of a natural order; this takes shape in the equations of monetary equilibrium, which in the sphere of monetary policy engenders the claim for stabilization of different variables. The other is the idea of interrelated processes in time; this idea does neither necessarily nor pertinently lead to the differential concept (the rate of infinitesimal increase), but to an analysis of processes of evolution and change and their interdependence. In the sphere of monetary policy such a point

[13] See, however, F. Machlup, *Börsenkredit, Industriekredit und Kapitalbildung*, Wien, 1931.

[14] Among recent studies, however, cf. J. Marschak, "Die Verkehrsgleichung," *Archiv für Sozialwissenschaft*, Band 52, 1924, which touches on this problem. Marschak's discussion of the consequences of a proportional expansion of credit and of the income velocity of circulation of money has subsequently been much developed by D. H. Robertson, R. F. Harrod, J. C. Gilbert and others.

[15] Walras is in this as in so many other respects astonishingly in advance of his time. Cf. *Éléments*, 1924 ed., 29 leçon. Cf. also A. W. Marget, "The Monetary Aspect of the Walrasian System," *Journ. of Polit. Economy*, 1935. Marget shows that Walras was a forerunner in the exposition of an equation of exchange, of the cash-balance concept, of the concept of forced saving and of velocity of circulation of goods, but first and last with respect to his idea to encompass monetary equilibrium in the general system of simultaneous equations.

of view will call forth measures of much more complicated and specified character than the actions of stabilization policies.

Starting from Wicksell's study of the relations between money-rates and prices,[16] we may point to a glaring dualism in this pioneer work. On the one hand Wicksell presents the result of a deductive-inductive analysis of the secular interdependence between money-rates and the price level and this factor's mutual relation to the real rate of interest, i.e., the average earnings of enterprise. It is a brilliant study which in some respects anticipates Schumpeter's theory of economic development. J. S. Mill was the first economist to suggest autonomous variations on the commodity side of the money function, but he did so by introducing a somewhat ambiguous velocity of circulation concept. Wicksell's leading idea is to show the occurrence of time changes in money and commodity factors, and thus to study the process of secular rhythm.[17] On the other hand, he suggests the conception of a normal rate of interest, "which keeps the price-level constant." The passages in Wicksell's study which bear on this idea are, however, by no means numerous;[18] it is only in the post-war discussion that it has been regarded by other economists as the main thesis. Wicksell himself never stressed the equilibrium element in the money function, and he attacked the belief in the stabilization of the price level.[19] That part of Wicksell's theory of money which is mainly based on an acceptance of the classical vocabulary has, however, been regarded as a theory of monetary equilibrium and has been pressed into the service of a policy of price stabilisation. Whilst Fanno[20] seems to have been the first to have grasped the true character of Wicksell's monetary theory, which is a description of a definite time process, the secular period, and the levels

[16] K. Wicksell, *Geldzins und Güterpreise*, Jena, 1898.

[17] J. F. Feilen, *Die Umlaufgeschwindigkeit des Geldes*, Berlin, 1923, p. 127, does not detect this superiority in Wicksell's reasoning in support of J. S. Mill. Cf. also L. Mises, *Theorie des Geldes und der Umlaufsmittel*, 2nd. ed., München, 1924, p. 98. This author has not grasped the time-process element in Wicksell's exposition. M. W. Holtrop in "Die Umlaufgeschwindigkeit des Geldes" in *Beiträge zur Geldtheorie*, F. A. Hayek, Editor, Wien, 1933, regards to a certain extent the velocity of circulation factor as the moving and driving variable, thus introducing the time-process element.

[18] J. G. Koopmans in "Zum Probleme des 'Neutralen' Geldes," *Beiträge zur Geldtheorie*, p. 273, has collected these passages.

[19] K. Wicksell, "Penningvärdets reglerande," *Ekonomisk Tidskrift*, 1913, and "Det definitiva ordnandet av världens penningväsen," *Nationalekonomiska Föreningens Förhandlingar*, 1925. Cf., also, Brinley Thomas, "The Monetary Doctrines of Professor Davidson," *Economic Journal*, 1935, where some aspects of Wicksell's theory are mentioned.

[20] M. Fanno, *La banche e il mercato monetario*, Rome, 1912.

governing this perpetual *transition*, Myrdal[21] and Hicks[22] have tried to reduce Wicksell's ideas to the classical and neo-classical setting, i.e., the determination of the conditions of monetary equilibrium which form the theoretical aim of Ricardo as well as of Cassel. The close relation between the idea of equilibrium and the determination of monetary policy, not the description of monetary phenomena, is here clearly, though indirectly, shown.

In Fisher's works we detect the same dualism.[23] On the one hand, we have the compound equation of exchange in which, indeed, the different magnitudes represent other numerous variables; the removal of all factors except one, the price-level, to the right side leads directly to an economic-political thesis, the plea for a compensated dollar. On the other hand, we have the cumulative process following from the differences between the real and nominal rate of interest, the changes in the purchasing power of money which gradually give the borrower an advantage during the upward swing of the cycle, and the lender a corresponding advantage during the depression. The time-process element is in fact markedly stressed in this demonstration, the rate of increase per unit of time and the psychological time-factor being symmetrically represented.[24]

Finally, in Keynes' treatise[25] we have, on the one hand, a fundamental equation expressing the price-level on the left side as determined by an identity on the right side, in which total money income, total output of goods, investment and saving are represented. This statement is introduced by the following sentence: "The fundamental problem of monetary theory is not merely to establish identities or statical equations relating the turnover of monetary instruments to the turnover of things traded for money. The real task of such a theory is to treat the problem dynamically, analysing the different elements involved in such a manner as to exhibit the causal process by which the

[21] G. Myrdal, "Om penningteoretisk jämvikt," *Ekonomisk Tidskrift*, 1931.

[22] J. R. Hicks, "Gleichgewicht und Konjunktur," *Zeitschrift für Nationalökonomie*, Band IV, this paper being mainly an attempt to show that the money market falls outside the general system of equilibrium, because the want of credit is founded on perfect knowledge of future events, correct anticipations being a necessary element in general equilibrium. Cf., also, *Economica*, 1934, p. 483.

[23] I. Fisher, *The Rate of Interest*, New York, 1907, *The Purchasing Power of Money*, New York, 1911, and "The Debt-Deflation Theory of Great Depressions," ECONOMETRICA, Vol. I.

[24] An attempt to introduce the notions of rate of increase and friction into the discussion of the six variables in Fisher's theory of exchange is found in Erling Petersen, *Den Moderne Kvantitetsteoris Gyldighet for Pengeverdiens Bestemmelse*, Oslo, 1933.

[25] J M. Keynes, *A Treatise on Money*. London, 1930, Books III and IV.

price-level is determined and the method of transition from one position of equilibrium to another."[26]

The monetary discussion of the last five years has, moreover, shown that the setting of the fundamental problem in this sphere is largely determined by two things, the actual economic conditions and the form in which the theory is presented. The rigorous formulation of equilibrium theories was based on the experiences of wartime inflation, viewed against the background of a classical natural order. Cassel's "purchasing power parity" and his plea for the stabilisation of the price-level is perhaps the most striking example, but there are others of greater scientific interest. Thus Lindahl's arguments[27] in favour of a regulation of the price level in inverse proportion to productivity are partly a program of stabilisation, partly a direct application of the Marshallian supply curve, i.e., a purely hypothetical concept in the method of variation. As a stabilisation program, however, it overlooks the fundamental factor in business cycles, the cumulative process of capitalization during the upward swing, changes in the price-level of consumption goods, which "should" vary inversely with productivity, being a secondary effect of that process. Hayek's intensely debated proposal regarding "neutral money," meaning the stabilisation of the volume of money, which has been traced to Wicksell's formulation of the quantity theory, is perhaps the most static of methods adopted in this sphere; the time-concept is the "imaginary" time in Böhm-Bawerk's theory of capital.[28]

In contrast to these theories, where statical methodology and the equilibrium element is stressed, we have Robertson's, *Banking Policy and the Price Level*, 1926, which, as the first post-war monetary investigation, logically develops the time-process element. Robertson sets out from views based on a thorough knowledge of the real variations in time and the theories of business cycles, especially those of Aftalion.[29] The latter theory may be regarded as an adaptation of the Austrian theory of value and of the Austrian theory of capital to the time-process principle. The variation in the valuation of goods in terms of

[26] *Op. cit.*, p. 133. See below on the implication of consecutive equilibria.

[27] E. Lindahl, *Penningpolitikens mål*, Malmö, 1929, and *Penningpolitikens Medel*, Malmö, 1930.

[28] Cf. contributions by Keynes, Hayek, Sraffa, G. Halm, Fanno, J. Åkerman, in *Economic Journal*, *Economica* and *Zeitschrift für Nationalökonomie*, 1930–34. Also E. F. M. Durbin, *Purchasing Power and Trade Depression*, London, 1933, and J. E. Meade, *The Rate of Interest in a Progressive State*, London, 1933.

[29] A. Aftalion, *Les Crises Périodiques de Surproduction*, Paris, 1913. Profit and profit-making as a central factor in the business cycle is not kept clearly in view in Aftalion's theory, which in this respect must be supplemented by the propositions of Veblen, Lescure and Mitchell.

money during the cycle and the average period of production of capital goods or instruments form the basis of this system.[30] An introductory study of "appropriate fluctuations of output" marks, in Robertson's work, from the very outset, a point of view in which no attempt is made to embrace all the variations in a state of equilibrium, and which a priori rejects the idea of perfect stabilisation. Finally, the various economic activities and especially the various modes of savings are described, and their interrelation and bearing on the business cycle discussed. It is evident, however, that the interconnexion among these various activities, each one covering an average time-space of its own, is of such an intricate nature that any attempt to sum them up in an equilibrium position or to describe their correlation at a moment of time is doomed to failure.

The theory of money, in so far as such a theory can be isolated from general economic theory, will presumably have to make a fresh start on the line suggested by Robertson. That this will be the case is, in some measure, already shown by the latest contributions,[31] and may indirectly be inferred from the great influence that Robertson's succinct essay has exercised, in spite of an unpractical method in the presentation of the motives of entrepreneurs.[32] Robertson has also treated the time premiss and, in some degree, the psychological premiss in a new suggestive way. The classification and averaging factor has at the same time been thoroughly investigated by G. Haberler and others.[33]

To sum up our review of the different statements of the central problem in discussions tending towards a partial modification of equilibrium theory, it may be established that in the domain of distribution as in that of money, all of them set out from the idea of a natural order which can be expressed by equations representing either an identity of concepts and sums of concepts, or a functional relation

[30] Economists of the Austrian school have sometimes charged business cycle investigators with a misrepresentation of Böhm-Bawerk's original theory, when the term period of production has been used to mean the average period of construction of capital goods, not the imaginary length of a roundabout way of production. Such a charge is based on the assumption that terminology must be kept unchanged through the ages and is, therefore, scarcely warranted.

[31] Cf. E. F. M. Durbin, *The Problem of Credit Policy*, London, 1935.

[32] This is the Cambridge variant of labour cost theory of value; it is of course awkward to have to reason on "disutility of effort." Other valuable recent treatises have had their meaning somewhat obscured in this way. Cf. K. S. Isles, *Wage Policy and the Price Level*, London, 1934.

[33] G. Haberler, *Der Sinn der Indexzahlen*, Tübingen, 1927. Cf. H. Staehle, "A Development of the Economics of Price Index-numbers," *Review of Economic Studies*, Vol. II, June, 1935, Ragnar Frisch, "Annual Survey on Economic Theory," ECONOMETRICA, January, 1936.

in which certain factors are labelled cause and other factors effect. The idea of interdependence is, however, gradually widened and deepened, but owing to the formal requirements involved, this extension in scope is gained at the expense of realism; the mathematical claim for as many equations as unknowns having nothing in common with the manifold processes in time which economic science endeavours to grasp and describe. Moreover the classical dualism of market value and normal value still makes itself strongly felt in a number of modern modifications of the original setting. Market value is a figure and remains so, but normal value branches out into normal price, which may be represented by an average, and true price, which leads to pure concepts such as labour cost and "disutility of effort." To the confusion arising from interrelation with time and from concepts with empirical data, we may finally add that classical starting point of arguments, the "first cause," which has a tendency to shift its meaning as rapidly as the movements of a search light. At one moment the first cause is the selected starting point in deductive reasoning along the line of the method of variation, at another moment it assumes the shape of the origin, the source, the *primum mobile* of an economic chain of events.[34]

4. THEORY OF RISK

The necessity of introducing "time" and "change" to supplement the thesis of economic identities has led to several partial modifications of equilibrium theory. In the degree that these modifications in the theory of distribution and of money have proved to be too limited in scope, general modifications have been made in the formulation of the theory of equilibrium. To the stationary and statically analysed system a formal change-factor has been added in the shape of a coefficient of risk. This coefficient is, however, a constant in the system of equilibrium. Consequently, in most analyses carried out along this line the coefficient of risk having been noted, the investigator declares that the acting parties on the market know what is going to happen to prices, supply, and demand in all times to come, "future is anticipated."

Though Jevons, Cairnes, and Marshall and doubtless many other

[34] Nowhere is this fact more obvious than in the neo-classic theory of international trade. Out of the three recently published elaborate treatises in this field by B. Ohlin, G. Haberler and C. Iversen, mentioned in the order of issue, the last named, *Aspects of the Theory of International Capital Movements*, Copenhagen and London, 1935, shows any reader who is alert to the setting of economic problems that the causes and effects in respect to the relation between international transfers of gold, capital, and goods, as well as cost and other production factors within the different countries, are entirely dependent on what link in the "chain of causation" is chosen as starting point or "first cause."

economists of the nineteenth century have indicated anticipations of future change in various economic factors, Pareto has been the first to approach this all embracing question systematically. His "courbes de poursuites" give an extremely suggestive analysis of the relation between consecutive actual change and consecutive actions called forth by these changes.[35]

The next step in advance was taken by F. H. Knight, whose work on risk and allied problems is one of the most stimulating treatises of modern economics.[36] Despite the author's views to the contrary, there is much substantial novelty in this investigation, this mainly in two respects. In studying the risks which give rise to profit, Knight has opened the discussion of the rôle played by anticipations in the determination of prices, and in defining the characteristic features of enterprise and free competition, he has made an important contribution to the formulation of the theory of monopoly. Starting from Pareto, Davenport, and Schumpeter, the author thus suggests two general formal modifications of equilibrium theory, one leading to a formal extension of the Walrasian theory of interdependent prices, the other to the modern formulation of the theory of business enterprise as a purely theoretical problem.

Following this line G. Myrdal generalises the conception of risk among groups of entrepreneurs.[37] Anticipations of future costs, prices, and profits, combine to determine the prices of products and the prices of means of production in a general functional system. These combined anticipations, however, must be analysed if we desire to distinguish their various components. First, we have to reckon with the objective risk which the entrepreneur actually incurs. Second, there is the conception or idea which he holds about this objective risk. Third, we must consider the valuation he puts upon the risk which he is incurring. Starting from Cassel's equations of price interdependence, the author attempts, by dint of an appreciation of individual risk, to introduce changes into the system, "thus delivering it from its statical premiss." Myrdal's study, in a greater degree than that of Knight, addresses itself to the single problem, the question of costs and prices and their general interdependence in the light of the conception of economic risk. This dynamic theory reckons merely with the oscillation

[35] V. Pareto, *Manuel d'Économie Politique*, Paris, 1909, p. 289. In a paper on "The Rôle of Time in Economic Theory," *Economica*, Feb. 1934, p. 92, also *Zeitschrift für Nationalökonomie*, Band I, p. 139, P. N. Rosenstein-Rodan adopts Pareto's exposition without remembering the source.

[36] F. W. Knight, *Risk, Uncertainly and Profit*, Boston, 1921, also No. 16 in a series of reprints published by the London School of Economics and Political Science, 1933.

[37] G. Myrdal, *Prisbildningsproblemet och Föränderligheten*, Uppsala, 1927.

of entrepreneurs' anticipations between the actual pricing process and one at an indefinite date in the future, but it does not follow the process along the road between these dates. An exposition of the dynamic problem from the standpoint of economic risk is indeed interesting and instructive, but it scarcely forms a suitable basis for a study of real variations, such as economic cycles. The reason for this statement is not that the theoretical premiss does not get beyond stationary equilibrium, but that the static methodology cannot be developed further when the momentary conception of risk is regarded as the kernel of the question.[38]

Starting from the setting of the problem proposed by Myrdal, E. Lindahl has attempted to bring the time-factor still further into the pricing-process regarded as a state of general equilibrium, by viewing it from the point of view of capital formation.[39] Just as monetary policy has been used to introduce autonomous changes in the sphere of deductive equilibrium, so capital, that factor of production which in some degree may be indentified with time, can naturally be used to the same end. It is assumed a priori that economic central theory may be identified with the theory of prices, Cassel's neo-classic formulation. Starting from the assumption that time has no economic relevancy, the setting of the problem is determined by the following premisses: all individuals have perfect knowledge of all actual prices, entrepreneurs seek to maximise their profits, no friction is reckoned with, i.e., full atomism, large and small scale production is equally remunerative, free competition prevails. The system of simultaneous equations, expressing general price interdependence, is given as the theoretical description of economic life in this timeless society, the working hypothesis behind the equations being that the supply of means of production, demand functions of consumers and technical coefficients are given data.[40] The problem is "solved" by dint of additional assumptions which tend to simplify it, viz., that the supply of and demand for productive services are equal, and that the price of each product equals the sum of the costs distributed as remuneration to productive services by its production. The next step must be to take some account of the time element. Individuals do not reckon only with the wants of the moment, the future is subject to change, the individual conception of future change takes the shape of guesses, in

[38] J. Åkerman, *Om det Ekonomiska Livets Rytmik*, Stockholm, 1928, p. 10.

[39] E. Lindahl, "Prisbildningsproblemets Uppläggning från Kapitalteoretisk Synpunkt," *Ekonomisk Tidskrift*, 1929.

[40] K. Wicksell, *Ekonomisk Tidskrift*, 1925, p. 111, has remarked that A. L. Bowley in *The Mathematical Groundwork of Economics*, Oxford, 1924, p. 51, though he does not reckon with a time-factor in his equations, still supposes that there is saving in the community. Lindahl has observed this discrepancy.

which the element of risk is included. Lindahl still assumes that individuals correctly anticipate future change, and also that the studied society is of a stationary type. In discussing certain elements in Böhm-Bawerk's explanation of interest on capital, namely the first, second and third "grounds," Lindahl touches on some time-elements, but we are once more reminded that Böhm-Bawerk's theory of capital has an imaginary time-factor. A further step in Lindahl's handling of the problem is to reckon with a nonstationary society, in which the factors of the pricing process are incessantly changing, whilst individuals are still in possession of perfectly correct anticipations of future change. We thus have a continuous change in the relation between different economic variables, but the author supposes that the position can be made clearer if the process is divided into short periods during which all prices are assumed to be constant. "The changes are thus relegated to the transitions between the different periods, but within each period the average condition is assumed to be stable."[41] This situation is expressed by a series of Walrasian equations referring to a series of points in time instead of to a single date. Finally, the author drops the assumption of perfectly correct knowledge of future change, but only to the extent that unexpected gains and losses are supposed to be arising in a community where all individuals have the same conception of risk and where this general conception is correct. The windfall gains and the unexpected losses are thus the shocks that shift the system from one state of equilibrium to another, from one set of simultaneous equations to another. At this point the author breaks off his analysis, he points out that the next step would be to introduce different inindividual conceptions of risk which represent each individual's ideas about the future as more or less probable alternatives. Lindahl's analysis, which has been working throughout with the assumption that future change is correctly anticipated, thus starts and terminates at the point from which Myrdal sets out in this study on the nature of economic risk.

The stream of thought which we are following made a new advance with a study by E. Schams on "comparative statics."[42] After an interesting analysis of nomenclature and methodology in which he points out that a deductive analysis of rates of increase with the method of differential calculus cannot be termed more dynamic than the common devices of the method of variation, the author characterises static economics as a theory of quantities in the domain of logic postulates. It is at variance with the laws of logic to introduce the time-factor from the very outset, as it is mathematically inconceivable to express

[41] *Op. Cit.*, p. 61.

[42] E. Schams, "Komparative Statik," *Zeitschrift für Nationalökonomie*, Band II.

exactly the simultaneous change of more than two interdependent variables. Only by the comparison of two sets of quantity relations divided by an interval can time-change be conceived. With such reasoning it is quite natural that Schams should point to Böhm-Bawerk's theory of capital as a perfect example of "comparative statics."

The views of this school, and a school it certainly is, are, finally, expounded in G. Mackenroth's treatise on the theoretical foundations of "price research" and price policy.[43] Here the different strands of this general formal modification of equilibrium theory are systematically unfolded. This statement of the problem may be briefly summed up as follows: (1) Prices include a priori the whole economic process. (2) Interdependence of prices in a stationary world is expressed by the Walras-Cassel system of equations. (3) This system is supplemented by coefficients representing periods of production and depreciation, and by the inclusion of time quality of goods as well as anticipation of prices. (4) An institutional premiss should be set up, but this requirement seems to be primarily intended as a warning against the acceptance of the implications of the theory of subjective value, instead of Cassel-Myrdal's value-free theory of price. (5) Theoretical results are obtained by criticism of concepts and by analysis of the logic of constellations of concepts.

Turning first to the theory of risk as a complement to the formulation of equilibrium economics, we may again ask: What kind of anticipations are meant when one assumes that the future is known? If the acting individuals actually knew everything about future costs and prices, they would, as O. Morgenstern points out,[44] be quite different sort of men in a different community, and they would undoubtedly act in another way than that supposed under the assumption of a foreseen future. Moreover, the "tatonnements" leading to Walrasian equilibrium would be quite unnecessary. Why seek to find an equilibrium price when it is determined by the common and unanimous, perfect foresight of all the individuals? The postulated "higgling of the market" preparatory to each deal does not come to an end by an intellectual process, but by a decision, an act of will. But, according to the premiss of perfect foresight, this act must also have been foreseen by the other parties on the market. Perfect foresight and economic equilibrium are thus a *contradictio in adjecto*.

But, as Marget has observed,[45] we may reckon with the actual social,

[43] G. Mackenroth, *Theoretische Grundlagen der Preisbildungsforschung und Preispolitik*, Berlin, 1933.

[44] O. Morgenstern, "Vollkommende Voraussicht und wirtschaftliches Gleichgewicht," *Zeitschrift für Nationalökonomie*, Band VI.

[45] A. W. Marget, "Morgenstern on the Methodology of Economic Forecasting," *Journal of Political Economy*, 1929, being a review of O. Morgenstern, *Wirt-*

technical, and commercial conditions of society as a series of obstacles artificially restraining the actions of bargaining individuals. Even if *A* has a perfect knowledge of the intentions of *B*, who in turn is at the same time correctly predicting the economic plan of *A*, inclusive of *A*'s knowledge of *B*'s insight and design, the fact remains that both *A* and *B* are restrained in their actions, and, therefore, also in their economic planning and in their anticipations, by the actual social and economic structure of society. *A* and *B* are not atomistic "economic men," they are individuals in a certain trade, with a certain income, and with a certain localisation in time and space. Morgenstern's criticism, which he also has applied to the problem of economic prognosis, seems to have overlooked the institutional premiss and the cumulative, irreversable nature of economic processes, i.e., one of the most common objections against equilibrium economics. And yet Morgenstern's remarks are well founded, if the accent is laid on averaging and classification and on the psychological premiss of the setting of the equilibrium problem as presented by Walras as well as by Cassel. There can be no common foresight, because such an assumption treats the acting parties as freely reckoning individuals and at the same time as atoms in a homogeneous population. On this ground the setting of the economic problem implied in the system of simultaneous price equations is open to criticism. Here we are brought back to fundamentals. Viewed from this angle, the open system of Ricardo with its residual claimant, which sounds peculiar in modern ears, is superior to the closed system of the Lausanne school. An open system can theoretically include individual action which, by averaging, becomes, or tends to become, a typical or average action; but a closed economic system of equilibrium is a statement of the same nature as the monetary equation of exchange; it enumerates some important factors and arranges them on the plausible assumption that sellers actually receive the sums that buyers pay. Such a system cannot describe activities which gradually transform the structure of society, and the addition of coefficients of risk or the concentration of change to transitions between perfectly immutable periods makes no difference in this respect.[46]

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schaftsprognose, Wien, 1928. Cf., also, B. M. Anderson, "Static Economics and Business Forecasting" in *Economic Essays in Honour of J. B. Clark*, New York, 1927.

[46] The problems discussed might, of course, be further illuminated by an analysis of their relation to a series of contributions to the question of dynamic economics. From the point of view of the history of economic thought, especially, R. Streller, *Statik und Dynamik in der Theoretischen Nationalökonomie*, Leipzig, 1926, and from the point of view of mathematical economics, R. Frisch, "Statikk og dynamikk i den økonomiske teori," *Nationaløkon Tidskrift*, Copenhagen, 1929. Among the large number of investigations on "statics and dy-

But behind this reasoning lies an idea of still greater significance affecting the mode of thinking along deductive and inductive lines, the relation between these lines gives the key to the problem. With Knight, and still more with Myrdal, Lindahl and Mackenroth, the reasoning is characterized by a rigid and absolute line of demarcation between concepts and data, between notions and manifestations. If one adopts Planck's differentiation between absolute causation in a typically constructed model-world and relative causation, founded on statistical measurements and laws of probability, the theorists of this school are strong partisans of the former system. Against "empiricism," "who observes and answers without questioning clearly and correlatively," they set their demand for a logical freedom of inner contradiction.[47] This demand, which can scarcely be challenged as a necessary prerequisite to an econometric inquiry of some import, is, however, so strongly stressed that any thought of communication between observations and economic theory seems to be inconceivable. The intuitively discovered laws of the classical economists should not be reconsidered by means of empirical tests, but by an inquiry into the logical correlation of the different laws in that system. Statistical investigations should be treated as purely statistical operations, in which the collection of data, their classification and averaging is the problem, econometric analysis and a closer coöperation between the principles of economics and statistics being regarded with scepticism.

It is of interest to observe that this intrinsic opinion is the result of a confluence of three different streams of thought, (1) the neo-Kantian school of philosophy ("Marburg school") with its marked differentiation between a metaphysical "thing-in-itself" and the phenomenal world, (2) the theory of Max Weber, with its criticism of "objective economics" and its line of demarcation between value and subjective valuation, (3) Ricardo's theory of value and Cassel's and Amonn's value-free theory of price. No wonder that such a flood should sweep away all the bridges between logical type-constructions and empiricism.

In spite of, or indeed perhaps thanks to, this intransigence, the school which we have discussed has unquestionably played a very important rôle in the development of modern economics. The pushing

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namics" the following may be quoted as specially bearing on the relation of value and time in a realistic setting: S. N. Patten, "The Theory of Dynamic Economics" in *Essays in Economic Theory*, New York, 1924, originally Univ. of Pennsylvania Law series, 1892; J. Schumpeter, *Theorie der Wirtschaftlichen Entwicklung*, München, 1911; and among recent papers, for example, J. Åkerman, "Dynamische Wertprobleme," *Zeitschrift für Nationalökonomie*, Band II, 1931, and O. Morgenstern, "Das Zeitmoment in der Wertlehre," *ibid.* Band V, 1934.

[47] G. Myrdal in *Ekonomisk Tidskrift*, 1931, p. 302.

of the logical requirements of the equilibrium system to its extremes has demonstrated that the time-element and the value-quantity relation are jointly and inherently involved in any attempt to widen the scope of equilibrium economics.

The theory of risk may in one sense be regarded as a *generalised theory of enterprise,* but in the construction of general equilibrium based on the individual's correct anticipations we find not only entrepreneurs but any relevant economic person who is a bearer of an anticipation. In order to get rid of the valuation-quantity relation and all the diffiulties inherent in a theory of value, the value-free theory of price has been introduced, notably by the same school as that of the adherents of the theory of risk.

The theory of enterprise, which is used here as an inclusive term covering the theories of cost as a problem of entrepreneur economy and the theory of monopoly, goes a more direct way to extricate itself from the entanglements of the valuation-quantity relation. This way, which was indicated by Marshall, leads to the "representative firm," which may be a marginal or typical enterprise. In this construction the psychological and institutional premisses are stated in purely liberalistic terms, whilst the time factors and constants premisses are treated on the model of the deductive theory of variations, and this originally, and most markedly, in the theories of decreasing and increasing return.

In the recent elaborate and lucid contributions to the theory of cost and to the theory of monopoly there seems to be a growing opinion that these problems should be treated either as pure theories of enterprise with its concepts of cost and price, or as a form of general economic theory with its different view of value, cost and price. In both theories a meaning may be imparted to marginal revenue and marginal costs as well as demand functions; if the investigation is carried out from start to finish within the limits of one of them, the result will not be ambiguous and may even serve both theories. In this sphere, the concept of opportunity cost and other matters relative to the labour cost theory of value are the worst obstacles to clear treatment and definite results. In the present paper on the setting of the central problem we must confine ourselves to these observations.

5. ECONOMETRIC ANALYSIS

What constitutes the step from formal to real modification of the theory of equilibrium as an all-inclusive economic theory? Empiricism starting from the idea that facts speak for themselves does not give the answer. The difference is to be found in the setting of the central economic problem. Formal modifications of equilibrium theory, and particularly the variation suggested by the "risk-theorists," find the

ideal of abstract description in the perfect logical circle, i.e., a closed deduction leaving no gaps where an analysis of change or causality can be introduced. The setting is thus a priori tautologous; it arrives at results which are exactly identical with the elements of thought which have been put into the argument. Such a theory sacrifices the observations of induction to the claims of logic accuracy. In this connexion a fundamental principle of economic theory may be formulated in the statement: what is gained in stringency is lost in realism. If the adherents of a purely logical theory of equilibrium, and especially the modern propounders of the conception of a state of general correct anticipations, were confronted with such a dictum, they would be bound to choose *perfect* stringency and *no* realism. In such a setting the new factors introduced into the system of simultanous equations, coefficients of risk, of friction, of velocity of reaction, of structual change, are not variables but constants, just as the Walrasian coefficients of production.

Let us now return to our introductory postulates: (1) Valuation-quantity relation, (2) Classification and averaging problem, (3) Institutional and structural premiss, (4) Psychological premiss, (5) Problem of constants, (6) Time-postulate. In a sense it may be said that the uncompromising theory of logical economic equilibrium has debarred itself from the possibility of even approaching these postulates, which form a series of interrogations and cannot, therefore, be reached from a set of absolute affirmations. How then should these interrogations be included from the very start in an economic investigation?

It is this question which has been the object of reflection since Malthus formulated his law of population on the basis of inductive intuition, and Jevons brought his scientific brain into contact with social problems. Over the biometricians, Galton, Pearson, and Yule, and the statistician, R. H. Hooker, this admirable series of English contributions to Baconian methodology leads to the American systematic attempts at econometric analysis. Mitchell's[48] and W. M. Persons'[49] presentation of the economic time-series, the calculation of various index numbers and of indices of seasonal variations and secular trend as well as the determination of time-lags, constitute the *first* important chapter in a partial real modification of the setting of economic theory. But this setting is remote from economic deduction as well as from a general structural analysis; out of our six postulates it includes only the classification and time premisses (2, 6), all others, and particularly the fundamental valuation-quantity relation, are omitted.

[48] W. C. Mitchell, *Business Cycles*, Berkeley, 1913.

[49] W. M. Persons, *Review of Economic Statistics*, Preliminary Vol. I, Harvard, 1919.

The *second* chapter is opened by H. L. Moore[50] and H. Schultz,[51] who seek to determine empirically the Marshallian laws of the relation between quantity and price. Here the time factor is left out, though it might, in some measure, have been brought under investigation by a study of the shift of demand curves through time. The elimination of the time scale involves the inclusion of the deductive setting of the problem of value, and the interesting discussion on the meaning of statistical demand curves has also thrown light on the *ceteris paribus* premiss. The valuation-quantity relation, the classification problem, and the problem of constants (1, 2, 5), are thus considered in these econometric analyses, whilst the remaining three are ignored.

Attempts to open a *third* chapter of econometric analysis have been made by C. F. Roos,[52] P. Douglas,[53] J. Tinbergen[54] and others, in which the connexion between the general valuation-quantity relation and the different market problems is omitted, but where these market analyses and statistical determination of production coefficients are searching and extend over an increasing area. Here the classification, the structural, the constants and the time-postulates (2, 3, 5, 6), are treated, the fundamental and coordinating valuation and psychological premisses being omitted.

More ambitious, however, are the notable endeavours of I. Fisher[55] and R. Frisch[56] to open a *fourth* chapter of econometric analysis, based on measurement of marginal utility, adopting an interlocal or intertemporal standard. These investigations boldly face the valuation-quantity relation, but, in opposition to the value-free price theory, should perhaps more explicitly state that we have to start from the following assumption: We can study the external world in its manifestations, but the difficulties arising from transcendental concepts of value are self-constructed difficulties blocking the way to the premisses of the central problem. The "economic principle" may, moreover, be stated in a form that does not permit of any obstacles arising on the

[50] H. L. Moore, *Laws of Wages*, New York, 1911.

[51] H. Schultz, *Statistical Laws of Demand and Supply*, Chicago, 1928. Cf., also, "Selected References on the Theoretical Aspects of Supply and Demand Curves and Related Subjects," ECONOMETRICA Vol. II, p. 399–421.

[52] C. F. Roos, *Dynamic Economics*, Bloomington, 1934.

[53] P. Douglas, *The Theory of Wages*, New York, 1934.

[54] J. Tinbergen, "Annual Survey: Suggestions on Quantitative Business Cycle Theory," ECONOMETRICA, Vol. III, and literature cited there.

[55] I. Fisher, "A Statistical Method for Measuring Marginal Utility," *Economic Essays in Honour of J. B. Clark*, New York, 1927.

[56] R. Frisch, *New Methods of Measuring Marginal Utility*, Tübingen, 1932. Cf. R. Frisch, "Sur un Problème d'Économie Politique Pure," *Norsk Matematisk Forenings Skrifter*, I: 16, 1926.

way to a constructive setting of the valuation-quantity problem. (1) If different alternative incitements are competing, the relatively strongest determines the decision. (2) The conception of utility of different galaxies of goods affects the decision of every person in such a way that he feels the utility of a certain galaxy as being the relatively strongest. (3) The person at issue must decide on that galaxy.[57] Fisher and Frisch approach the problem in such a spirit, though the introductory treatment of the subjacent theory of knowledge might be further extended. Out of our six postulates all but the sixth are reckoned with, the time-postulate being treated as a constant in order to give the result a direct meaning in the time-free general theory of economic equilibrium.

In this summary of the problems of econometric analysis we have addressed ourselves only to one single point, the setting of the central problem, and this in a systematic scheme which undoubtedly is much too condensed. We have merely underlined the fact that the four parts of econometric analysis have so far been confined to a *partial*, though real, modification of equilibrium theory. Is a general real modification possible, that is, a theoretical description which includes from the outset all the six postulates? If so, how is the fundamental question to be formulated?

6. ECONOMETRIC SYNTHESIS

The gist of the preceding exposition has been an attempt to show that the "dynamic element," the "time-factor," the "real change along the time-scale," cannot be introduced as an addendum to a pure theory of equilibrium. From this point of view nothing can be more fundamentally "wrong" (i.e., in principle, blocking the way to theoretical development) than Böhm-Bawerk's dictum to the effect that the business-cycle theory must necessarily be the last chapter in a generally accepted theory of economic equilibrium.[58] The element of change cannot be brought in as a second thought; the result will be analogous to that experience of the baker who, having forgot to put yeast in the dough, threw a handful of it into the oven. If we attempt to discuss the time element in the theory of capital, the partial disequilibrium in the theory of money, the risk and anticipation elements, and the veloc-

[57] H. Bernadelli, *Die Grundlagen der Ökonomischen Theorie*, Tübingen, 1933, p. 13. Cf. the meaning of Pareto's "ophélimité," *op. cit.*

[58] The fact that Böhm-Bawerk's theory of the roundabout way of production, though not necessarily to be regarded as an integral part of central political economy, has been the most powerful lever in the development of business cycle theory merely shows that the history of economic doctrines does not follow principles outlined in advance. The same may, to a certain extent, be true of Hayek's production triangle, in which the x-axis denotes consumption-goods and the y-axis complementary goods, *and "time" as well.*

ity of reaction within the boundary of a strict identity between two sets of variables, and imagine that by such a discussion we have described the real time changes and their interdependence, we shall certainly be deceived.

We have seen that partial real modification of equilibrium theory cannot be converted into a general setting of the problem because (1) the valuation-quantity relation and (6) the time-postulate involve the inclusion of alternatives which apparently cannot be reconciled within the same framework. A starting point where the processes in time and the valuation elements are *symmetrically* and *simultaneously* included seems to be the only possibility left.[59] We have then to reckon with an element of time, an element of volition, which proves to be identical with the psychological premiss, and an element of valuation. We must moreover observe the structural premiss, the *ceteris paribus* postulate and the inductive classification problem, but these latter postulates do not present great difficulties when the former are settled. The key to such a treatment must be two concepts, *activity* and *period*, and their addition in one unit. *Periods of specified activity* are thus to be regarded as the atoms of econometric synthesis.

In such a construction everything depends on the choice of standard periods and on the correlation of period and activity, i.e., the process of valuation stretching over the same space of time as the measurable period of quantitive change. As regards these primary *activity-periods* we may state that they are:

(1) *integral*—the idea of a moving equilibrium, representing a total trend covering all periods, is therefore superficial, and moreover ignores the interdependence of different activity-periods;[60]

(2) *perpetually operating*—there are *no* bygones in the economic mechanism, every movement being connected with an eternal previous sequence of variations;[61]

(3) *starting at different points in time*—the concept of maintained equilibrium at a point of time as well as through a period is inconceivable, a cut through the income-stream showing activities in different phases of increase or decrease, and moving at different speeds, subsequent cuts moreover showing quite different proportions between periods;[62]

[59] The author discusses these problems in a treatise, *Ekonomisk Kausalitet*, Lund, 1936.

[60] This is perhaps the chief objection to H. L. Moore, *Synthetic Economics*, New York, 1926. Cf. Roos, *Dynamic Economics*, p. 8.

[61] "Bygones are bygones in business" may be good enough as a psychological postulate in the theory of enterprise, but does not suffice in a general explanation of economic change through time.

[62] H. Mayer, "Der Erkenntniswert der funktionellen Preistheorien," *Wirtschaftstheorie der Gegenwart, II*, Wien, 1932, p. 198, says that in the Walrasian system only the *final* phase of the whole process is investigated, i.e., only the price-equilibrium as a result of given prices.

(4) *irreversible*—as the activity-period covers a historical process, it can only move with time, the cumulative movement of short periods being subsumed in longer and longer periods, but no subdivision of periods being logically possible;[63]
(5) *of average length*—the year being the natural unit of time[64] covering the activity of wholesale and retail trade as well as of agriculture, the business cycle being the activity period of industrial capital formation, and a secular wave covering the activity of a generation's saving and of State and communal capital formation;
(6) *changing gradually as to quality and scope*—the institutional and structural mutation working on the mechanism in a way that can sometimes be explained but can rarely be expressed quantitatively.[65]

Econometric synthesis working from some such assumptions may perhaps succeed in formulating a general real modification of equilibrium theory by starting from the observation of activity-periods, thus including the valuation-quantity relation *and* the time element together with the four other factors named among our postulates. If such is the case, this may be due to the fact that development of economic science "does not consist in the transference of scientific attention from one set of concrete phenomena to another set of concrete phenomena; but rather in the adoption of a different *attitude* towards the asking of a different set of questions about the same problem."[66] Here again Walras, writing at the eve of this century, anticipated the actual need, when he spoke of the necessity of a symbiosis between philosophy, history, economics and mathematics.[67]

A review of recent contributions to economic theory tends to show—if attention is concentrated on the setting of the central problem, that in economics the time has not come to speak of many things, the time has come to speak of fundamental things.

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[63] This, of course, is the gist of the Austrian theory of value, Menger, and of capital, Böhm-Bawerk.

[64] A. A. Cournot, *Recherches sur les Principes Mathématiques de la Théorie des Richesses*, Paris, 1838.

[65] F. H. Knight, "Statik und Dynamik," *Zeitschrift für Nationalökonomie*, Band II, p. 7, objects to the use of mechanical analogies in economics, because of the alleged impossibility of including evolutional categories. Economic science, however, cannot develop without symbols and analogies, and mechanical metaphors seem often to be superior to all others in applicability and visualizing force.

[66] R. W. Souter, *Prolegomena to Relativity Economics*, New York, 1933, p. 163.

[67] L. Walras, *Élements d'Économie Politique Pure*, Paris, 1926, preface to 4th edition, p. XX.

THE MARGINAL EFFICACY OF A PRODUCTIVE FACTOR

FIRST REPORT OF THE ECONOMETRICA COMMITTEE ON SOURCE MATERIALS FOR QUANTITATIVE PRODUCTION STUDIES

By E. H. Phelps Brown

Chairman of the Committee

The Econometrica Committee on Source Materials for Quantitative Production Studies was set up during the Paris meeting of the Econometric Society in 1932 on the initiative of the Editor of Econometrica. The Committee decided first to concentrate on collecting and presenting some *samples* of statistical and other numerical data that illustrate how the principles involved in the theory of production work in practice. At present, three reports on this subject are ready in manuscript covering the following topics:[1]

I. The marginal efficacy of a productive factor.
II. Cost categories and the total cost function.
III. The profit experience of producers and their response to price.

It is hoped that the samples given in these reports may be useful both as classroom examples in connection with courses in production and as suggestions for research workers initiating further investigations in this essentially econometric field.

For whatever interest the reader may find in this paper, he is indebted to the members of the Committee, whose task it has been to search out the materials of each country represented; but while co-operation has been general, it is proper to give special recognition to those who have transmitted the material now set out: to Professor Zeuthen of Copenhagen, for Denmark; to M. Georges Lutfalla, Actuaire-Contrôleur Adjoint au Ministère de Travail, for France; to Dr. Oskar Lange of Krakow, for Poland; and to Dr. Louis Bean, of the U. S. Department of Agriculture, for the United States. The services of those who helped the national collaborators will be found recorded among the footnotes to the text.

When the inquiry began, it was surprising to find so little ready to hand. There are some aspects of Supply that can be studied statistically more easily than anything in Demand, and yet it seemed at first as if statistical care had been spent upon Demand alone. Suppose the author of a work on Principles challenged by a sceptic holding that

[1] All these reports will appear in early issues of Econometrica. Because of the great amount of material on hand, it has not been possible to arrange for a more speedy publication of the reports.—Editor.

the law of diminishing return was of no consequence within the intensity of cultivation commonly found today, where could he lay his hand upon such facts as alone could resolve the issue?[2] We are concerned much, and rightly, with the theory of marginal productivity; but do we know, and is it not right that we should wish to know, where we can find some functions of marginal productivity statistically isolated and charted? To point after point in Supply might such questions have been directed; some of them could have been answered (and some of the sources mentioned in this paper are already well known); but for the most part there was no answer ready.

Yet as the inquiry went on, the matter for surprise soon became how much material there was, and how much use might be made of it, whether to test the empirical postulates of theory, to draw out new postulates, or to find which of the possibilities revealed to speculation apply most closely to the particular case. Considering the range of the material, the present writer concluded that his task must be not to compile a catalogue but to display some samples.

There must, however, be many who with regret will see how much that should be known has gone unsampled, and to these an appeal is addressed: if by their kindness the writer's ignorance may be repaired, so that later the gaps may be filled, a chief purpose of this publication will have been served.

THE MARGINAL EFFICACY OF A PRODUCTIVE FACTOR

THERE are good reasons to suppose that the contribution of one productive factor can more easily be isolated in agriculture than in industry. The technique of industry must often be such as to prevent our varying one factor at a time; there may be only one proportionate combination of factors that is feasible, or, though different combinations can be used, the change from one to another cannot be made by incremental changes in one factor alone. Even where variation is possible, its effects may be hard to distinguish, for the effect may be not so much to change the output of one product as to substitute one kind of product for another; or it may make itself felt only in the internal economies of the long run. In agriculture these difficulties may be absent. Crops may be grown and animals fed with different combinations of factors; variations may be made by small increments of one factor; changes in the quality of the product may not be so great as to prevent our expressing different outputs in a common physical unit; and the effects of variation may be directly apparent in the changes of the output thus measured.

[2] In Marshall's *Principles*, it is true, he would find a table showing the effects on yield of different amounts of ploughing and harrowing, recorded by the Arkansas Experimental Station and reported in the London *Times* of 1889.

It is not surprising, therefore, that all the material to be presented in this section relates to agriculture; yet we must not overestimate the experimental validity of the results. Where organic growth is to be studied, it is hard even in the laboratory to be sure of varying only one factor at a time; and the experiments which relate most closely to the economic problems of agriculture are those performed not in the laboratory but in farmyard and field, where a wide range of conditions escapes experimental control: "the yield of plots of about one-fortieth of an acre frequently vary among themselves owing to soil heterogeneity with a standard deviation higher than 10 per cent of the mean yield."[3] To meet these difficulties, the agricultural scientist has devised a special technique, both for the design of field experiments and for the statistical analysis of results.[4] This technique is here represented by the examples drawn from the recent work of the Rothamsted Station; in considering the other material, we must bear in mind the allowance to be made for *ceteris non paribus*.

(A) The prominent place which the law of diminishing returns has taken in the principles of Political Economy gives a special interest to the suggestions which have been made for a general formula of plant growth.[5] (a) The first generalization is perhaps that of Liebig,[6] "the crops on a field diminish or increase in exact proportion to the diminution or increase of the mineral substances conveyed to it in manure;" to which he added what came to be known as the Law of the Minimum, "by the deficiency or absence of *one* necessary constituent, all the others being present, the soil is rendered barren for all those crops to the life of which *that one* constituent is indispensable." (b) Later experimental work suggested that the curve of returns is not linear, as Liebig had held, but S-shaped; each equal increment of input effects at first a greater increase of output than its predecessor, but after a phase of proportionality the increments of output begin to diminish. The results of fifty-six years of experiment at Rothamsted,[7] given in

[3] Dr. R. A. Fisher, in "The Technique of Field Experiments," Rothamsted Conferences, XIII (Rothamsted Experimental Station, Harpenden, Hertfordshire, England).

[4] This work is notably connected with the names of Dr. R. A. Fisher and the Rothamsted Experimental Station. See Dr. Fisher's *Statistical Methods for Research Workers*, the reference given in note 3, and the account given in the Report of the Rothamsted Station for 1927–28, p. 36.

[5] The abstract which follows is based upon the account given by Sir E. John Russell, D.Sc., F.R.S., Director of the Rothamsted Station, in his *Soil Conditions and Plant Growth*, 6th ed., Longmans, 1932. I am very much indebted to Sir John and to Mr. F. Yates, B.A., Chief Statistician of the Rothamsted Station, for guidance and material generously provided.

[6] Liebig, *Chemistry in its Application to Agriculture and Physiology*, 3rd ed., 1843; here quoted from Russell, *op. cit.* p. 17.

[7] Russell, *op. cit.*, p. 131.

Table I, agree with this suggestion. (c) E. A. Mitscherlich[8] developed a formula upon the hypothesis that increase in output was proportional, not to the amount of the variable factor momentarily applied,

TABLE I

ROTHAMSTED STATION, ENGLAND. BROADBALK WHEATFIELD, AVERAGE YIELD FIFTY-SIX YEARS, 1852–1907

	Plot 5	Plot 6	Plot 7	Plot 8
Nitrogen supplied in manure, lbs. per acre	—	43	86	129
Total produce (straw and grain) lbs. per acre	2,315	3,948	5,833	7,005
Increase for each 43 lbs. nitrogen	—	1,633	1,885	1,172

but to the difference between output momentarily obtained and the maximum output obtainable with an unlimited application of the variable factor. We have, then,

$$\frac{dy}{dx} = (A - y)c,$$

where y is the output, x the input of the variable factor, A the maximum yield, and c a constant. Integrating, and assuming $y=0$ for $x=0$, we have

$$y = A(1 - e^{-cx}).$$

This formula Mitscherlich later modified to take account of the fact that an excessive input of one factor is injurious to growth; he assumed that this injurious influence was present throughout, and that its effect upon the momentary proportionate increase in output was directly proportional to the momentary input. The proportionate increase is given by

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{(A - y)c}{y}.$$

Representing the injurious influence by the term $2kx$, we then have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{(A - y)c}{y} - 2kx,$$

$$y = (1 - 10^{-cx})10^{-kx^2}.$$

This formula finds no place for a possible condition of increasing return

[8] E. A. Mitscherlich, "Das Gesetz des Minimums und das Gesetz des abnehmenden Bodenertrages," *Landw. Jahrb.* XXXVIII, 1909, 537–552. *Bodenkunde für Land- und Forstwirte,"* 4th ed., 1923. *Die Bestimmung des Düngerbedürfnisses des Bodens*, Berlin, 1925.

in the earlier stages of input, and it assumes that the relative change in output effected by a change in the input of one factor is independent of the attendant inputs of other factors. There may be cases in which the latter assumption is justified, but its general applicability has been denied.[9] As an example of results in conflict with it, we cite in Table II certain findings of Dr. Fisher in his analysis of the yield of wheat from 1852 to 1918 in the Broadbalk field at Rothamsted.[10] The yield

TABLE II
YIELD OF GRAIN, 1852–1918, ON CERTAIN PLOTS AT ROTHAMSTED

Plot no.	Manure	Mean yield, bushels per acre
3 & 4	None	12.27 ± .39
10	Ammonia	19.50 ± .83
5	Minerals	14.18 ± .44
7	Ammonia and minerals	31.35 ± .90

on plot 10 bears to that on plots 3 and 4 the ratio 1.589; that on plot 7 bears to that on plot 5 the ratio 2.211; the return to nitrogen is, therefore, greater in the presence of minerals. Further examples of such interaction will be given under (C). (d) A relation suggested by E. J. Maskell[11] and known as the Resistance Formula has been tested at Rothamsted by Bhai Balmukand[12] and R. J. Kalamkar.[13] The relation is of the form:

$$\frac{1}{y} = F(N) + F'(K) + F''(P) + \cdots + c,$$

where y is the yield, $N, K, P, \ldots$ are the inputs of the several factors, and the functions F, F'', F''', are given by

$$\frac{a_n}{n+N}, \quad \frac{a_k}{k+K}, \quad \cdots,$$

where a_n, a_k, are constants, and $n, k \ldots$ are the amounts of the several factors present at the outset in the untreated soil. With this

[9] See Briggs, "Plant Yield and Intensity of External Factors," *Ann. Bot.* CLV, 1925, 475–502.

[10] R. A. Fisher, "An Examination of the Yield of Dressed Grain from Broadbalk," *Journ. Agric. Sci.* XI, 1921, 107–135.

[11] E. J. Maskell, thesis deposited in Cambridge Univ. Library; *Proc. Roy. Soc.*, 1928, B, 102, 467 and 488.

[12] Bhai Balmukand, "The Relation between Yield and Soil Nutrients," *Journ. Agr. Sci.* XVIII, 602.

[13] R. J. Kalamkar, "An Application of the Resistance Formula to Potato Data," *Journ. Agr. Sci.* XX, 440.

formula, if y and y' be the outputs corresponding to the inputs X and X' of one factor, then, instead of the ratio y/y' being the same for all inputs of associated factors, as it is in Mitscherlich's formula, we now assume the difference $1/y-1/y'$ to remain the same. The parameters applied to each factor are still independent of the inputs of associated factors, but the marginal productivity function for one factor is not thus independent; for the case of two factors, it is of the form

$$\frac{\delta y}{\delta N} = \frac{a_n}{(n+N)^2} \div \left[\frac{a_n}{n+N} + \frac{a_k}{k+K}\right]^2.$$

This relation may be illustrated by the following data and computed results from Balmukand's paper (Table III):

TABLE III

AVERAGE YIELD OF POTATOES, TONS PER ACRE, ON GROUND UNIFORMLY TREATED WITH TEN TONS DUNG PER ACRE

(Data from Seale-Hayne Agricultural College, Devonshire, England)

Cwt. per acre, sulphate of potash	Cwt. per acre, sulphate of ammonia			
	0	1	2	3
0	4.85	6.18	7.38	7.37
1	5.19	6.72	7.82	8.23
2	5.46	6.81	7.37	7.95
3	5.79	7.01	7.85	9.00

Values of constants, computed by Balmukand:

$k=1.13\pm0.58$, $n=1.66\pm0.40$,

$a_k=0.0258\pm0.0170$ $a_n=0.1689\pm0.0480$.

TABLE IV

MARGINAL PRODUCTIVITY OF SULPHATE OF AMMONIA IN POTATO-GROWING, AT DIFFERENT LEVELS OF INPUT OF POTASSIUM, FROM MASKELL'S FORMULA FITTED BY BALMUKAND

Input of sulphate of ammonia, cwts. per acre	Marginal productivity is expressed in tons potatoes per cwt. sulphate ammonia per acre				
	Input of sulphate of potash, cwts. per acre				
	0	1	2	3	4
0	3.949	4.728	5.067	5.256	5.378
1	3.203	4.176	4.638	4.908	5.084
2	2.650	3.715	4.262	4.593	4.814
3	2.229	3.326	3.930	4.308	4.566
4	1.900	2.995	3.635	4.048	4.336

From the values given for the constants we may compute the marginal productivity of sulphate of ammonia at different levels of input of potassium, and we reach the results given in Table IV and illustrated in Figure 1.

It will be observed that the formula has been used to carry the computation by one increment in each fertilizer beyond the range of the

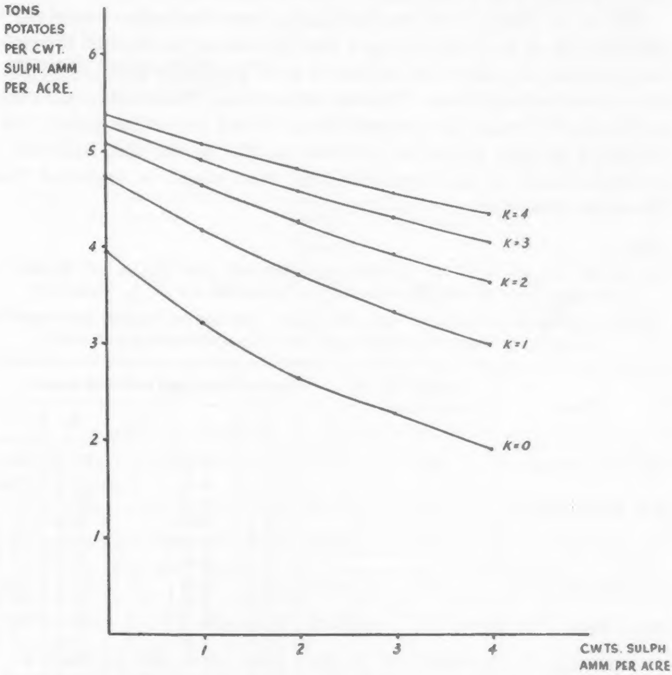

FIGURE 1.—Marginal productivity of sulphate of ammonia in potato-growing, at different levels of input of potassium, from Maskell's formula fitted by Balmukand.

original observations. Since, however, the formula does not reduce the marginal product to zero except as the input approaches infinity, it can be regarded as applicable only within a limited range of variation. (e) Exact knowledge has yet to be obtained of the form or forms of the functions within that range where variations in the input of one factor effect variations in output; but there will be general agreement that the marginal product does not gradually descend to zero, but at

a certain point begins to diminish sharply, and then disappears; the next increments of input add nothing to the product, and further increments may effect an actual decrease.

We may now present some of the statistical material which has been gathered. In section (B) will be found examples in which no study is made of interactions between factors, while in section (C) every example provides an illustration of such interaction.

(B)(1). In Table V will be found data from Australian experiments, recorded by J. A. Prescott,[14] and used by him to fit the first Mitscherlich formula, in order to calculate the most profitable input of fertilizer for a given price of wheat. Prescott reports that "reasonable agreement is obtained between the theoretical curve and the actual yields, but a tendency may be noted for the increase due to the first half-cwt. of superphosphate to give results higher than might be expected from the subsequent behaviour of the curve."

TABLE V

RELATION BETWEEN INPUT OF SUPERPHOSPHATE AND YIELD OF WHEAT IN CERTAIN AUSTRALIAN EXPERIMENTS REPORTED BY J. A. PRESCOTT

(Yield is given in bushels per acre; the lower row for each place expresses the marginal product in bushels per cwt. superphosphate per acre)

Place	Length of experiment, years	Superphosphate, cwt. per acre.					
		0	½	1	1½	2	3
S.A. Hd. Butler	10	10.0	15.7	17.7		20.9	21.6
			11.4	4.0		3.2	0.7
S.A. Booboorowie	10	21.0	26.2	28.5		30.5	31.6
			10.4	4.6		2.0	1.1
S.A. Minnipa	8	10.8	14.9	15.9		16.9	17.6
			8.2	2.0		1.0	0.7
S.A. Veitch's Well	7	15.8	16.5	19.0		21.6	21.0
			1.4	5.0		2.6	−0.6
S.A. Yurgo	4	12.6	23.8	26.2		30.6	31.0
			22.4	4.8		4.4	0.4
Vic. Werribee	12	7.8	14.8	17.2	19.4	19.8	
			14.0	4.8	4.4	0.8	
Vic. Longerenong	11	29.7	33.9	35.1		37.2	
			8.4	2.4		2.1	
Vic. Warracknabeal	8	13.4	19.3	20.9	22.0		
			11.8	3.2	2.2		
S.A. Roseworthy	9	16.0	19.9	20.1		21.4	21.4
			7.8	0.4		1.3	0.0
S.A. Roseworthy	18	14.9		18.5		20.7	
				3.6		2.2	
Vic. Salisbury	5	40.0	43.7	44.3	44.6	45.1	
			7.4	1.2	0.6	1.0	

[14] J. A. Prescott, Waite Agricultural Research Institute, University of Adelaide, in *The Economic Record*, Melbourne University Press, May 1928.

(2) From Danish experiments we have the following results (Table VI), cited by Professor Jens Warming.[15]

TABLE VI

THE EFFICACY OF NITRATE OF SODA, FROM DANISH EXPERIMENTS

(a) Barley

	Input of nitrate of soda in kg. per ha.			
	0	100	200	300
Yield, in hkg. grain	24	26	27	27.7
Marginal product, in hkg. grain per 100 kg. nitrate per ha.		2	1	0.7

(b) Beet

Input of nitrate of soda in kg. per ha.	Yield in hkg. dry matter per ha.	Cost of production in Kr.		
		per ha.	per hkg. dry matter	
			Average	Marginal
0	99	700	7.07	—
200	108*	748	6.93	5.33
400	114	796	6.98	8.00
600	118	844	7.15	12.00

* Figure adjusted by Professor Warming; in original record 105.

(3) The effect of fertilizer on beet is again illustrated in Table VII, which is an example of the experimental methods of the Rothamsted Station.[16]

(4) From a German source[17] are taken the figures displayed in Table VIII, showing the response of barley to nitrogenous fertilizer.

(5) We may add finally an example drawn from animal nutrition, though here the experimenters draw our attention to the presence of interaction. In this experiment,[18] three lots of Large White pigs, nine

[15] Jens Warming, *Landbrugets Graense-Kalkulationer*, G. E. C. Gad, Copenhagen, 1933. The data for barley come from P. Christensen, in an article in *Ugeskrift for Landmaend*, 1928, p. 371. Those for beet come from L. Rasmussen, *Beretning om Landboforeningerns Virksomhed paa Sjaelland 1926*, Copenhagen, 1927, p. 316. I am indebted to Professor Warming for generous assistance in the interpretation of his material.

[16] Rothamsted Experimental Station, Harpenden, England, *Annual Report for 1934*, p. 192.

[17] Weigert und Fürst, in *Zeit. f. Pflanz. Düng*, VIII 1929, 369; here taken from Sir John Russell, *Soil Conditions and Plant Growth*, 6th ed., p. 69.

[18] The experiment was performed at the Harper Adams Pig Feeding Experimental Station (Newport, Shropshire, England), and reported by the Principal, Charles Crowther, M.A., Ph.D., in the *Journal of the Royal Agricultural Society of England*, XCII, 1932, 33.

TABLE VII

SUGAR BEET, F. BELL, ESQ., MARKHAM MOOR, NOTTS, 1933
J. MCCLOY, ESQ., SECOND LINCOLNSHIRE SUGAR CO., BRIGG, LINCS.

Fertilizer	Roots (Washed)		Tops		Sugar Percentage		Total Sugar	
	Tons p.a.	*Increase*	Tons p.a.	*Increase*		*Increase*	Cwt. p.a.	*Increase*
Mean	*6.90*		*3.74*		*21.68*		*29.9*	
4 cwt.	6.74		3.18		21.80		29.4	
8 cwt.	6.98	*+0.24*	3.66	*+0.48*	21.80	*0.00*	30.4	*+1.0*
12 cwt.	7.04	*+0.06*	3.78	*+0.12*	22.10	*+0.30*	31.1	*+0.7*
16 cwt.	6.86	*−0.18*	4.35	*+0.57*	21.00	*−1.10*	28.8	*−2.3*
St. Error	±0.207	*±0.293*	±0.387	*±0.547*	±0.250	*±0.354*		

4×4 Latin Square. Plots: 1/50 acre.

Treatments: 4 levels of a complete fertiliser of the following analysis: N, 5%; water soluble P_2O_5, 5.7%; insoluble P_2O_5, 0.7%; K_2O, 10%.

Basal manuring: 12 loads of farmyard manure per acre ploughed in in winter.

Soil: Poor sand on gravel. Variety: Klein N. English. Manures applied: April 5th. Beet sown: April 25th. Lifted: September 18th. Previous crop: Barley.

Standard Errors per plot: roots: ±0.414 tons per acre or ±6.01%; tops: ±0.774 tons per acre or ±20.69%; sugar percentage: ±0.500. Mean dirt tare: 0.0536.

Conclusions.—The tops, but not the roots, show a significant response to increasing dressings of fertilizer, the sugar percentage a depression, barely significant, with the highest dressing.

TABLE VIII

YIELD AND PROTEIN CONTENT OF BARLEY RECEIVING VARIOUS QUANTITIES OF NITROGENOUS FERTILIZER

		N supplied, kg. per ha.				
		0	25	40	55	70
Yield of grain, dz. per ha.	1924	15.61	18.75	21.04	23.74	—
	1925	25.28	29.13	31.26	32.72	34.24
	1926	20.94	27.14	28.90	31.46	34.67
	1927	24.04	31.54	35.80	38.85	42.43
	1928	18.79	25.80	30.66	34.49	37.27
	Average	20.93	26.47	29.53	32.25	37.15
"Rohprotein" (N×6.25) mean per cent in dry matter of grain	1925 to 1928	10.78	10.12	10.37	—	10.93

in each lot, were taken, and to each lot was given the same dry ration of meals and minerals. Lot A was given no milk; Lot B was given one pint of separated milk per pig per day, and Lot C was given two pints. The results are recorded in Table IX. It will be noticed that the pigs receiving milk ate more dry food than did those without milk.

TABLE IX
EFFECT OF SEPARATED MILK ON THE GROWTH OF PIGS

Average Per Pig	Lot A lb.	Lot B lb.	Lot C lb.
Initial weight	46.6	47.9	49.5
Live weight gain	161.0	182.5	197.5
Carcass weight	156.8*	179.2†	192.3
Meal consumed	633	697	751
Milk consumed		147 pints	294 pints
	s.d.	s.d.	s.d.
Value realized at 6d. per lb. carcass weight	78 5	89 7	96 2
Cost of meal at 7s. cwt.	39 7	43 7	46 11
Surplus realized	38 10	46 0	49 3
Marginal product of milk, in pence per pint‡		0.585	0.265

* Estimated from results with 5 pigs sold, 4 unsold.
† Estimated from results with 8 pigs sold, 1 unsold.
‡ This entry is inserted by the present writer.

(C) (1) The efficacy of milk and grain in feeding pigs has also been studied by Professor Jens Warming[19] (Table X). In columns (1) and (2) will be found experimental data, which Professor Warming has taken from Report 128 of the Danish Laboratorium for landøkonomiske Forsøg, drawn up by Professor J. Jespersen; the grain used was half barley and half maize. These data Professor Warming analyzes to obtain the marginal productivities of milk and grain when different proportionate combinations of the two foods are employed. His calculations proceed on the assumption that the productive function underlying the data is homogeneous and of the first degree; if m and g be the quantities respectively of milk and grain used, and p the quantity of product, so that

$$f(m, g) = p,$$

then we shall have

$$f(\lambda m, \lambda g) = \lambda p,$$

where λ is arbitrary. On this assumption we are free, if we wish to facilitate comparisons, to multiply through the amounts of the productive factors and the product by the same constant; and since a given amount of product can be obtained from any proportionate combination of productive factors, we can study the effect of changes in this *proportion* between the factors engaged, without reference to the absolute level of intake and output. We can then in the first place

[19] Warming, *op. cit.*, p. 81.

TABLE X

MARGINAL EFFICACY OF MILK AND GRAIN IN FEEDING PIGS: RESULTS OF AN EXPERIMENT RECORDED BY PROFESSOR JESPERSON, AND ANALYZED BY PROFESSOR WARMING

	For 1 additional kg. of growth is used		Adjusted figures		Marginal productivity of grain as relative to marginal productivity of milk
	milk kg.	grain kg.	milk kg.	grain kg.	
	(1)	(2)	(3)	(4)	(5)
A	5.807	2.905	5.8	2.8	
					6
B	4.510	3.009	4.6	3.0	
					4
C	3.425	3.426	3.4	3.3	
					2
D	2.047	4.088	2.0	4.0	
					1
E	—	6.332	—	6.0	

	100,000 kg. milk with gain kg.	gives increase of kg.	Addition of grain kg.	gives increase of kg.	100 kg. more grain gives increase of kg.	100 kg. more milk gives increase of kg.
	(6)	(7)	(8)	(9)	(10)	(11)
A	48,300	17,200				
			17,000	4,500	26	4.3
B	65,300	21,700				
			31,700	7,700	24	6.0
C	97,000	29,400				
			103,000	20,600	20	10.0
D	200,000	50,000				
					16	16.0
E	—	—	—	—		

readily compare the marginal productivity of milk with that of grain. Thus, comparing rows A and B, columns (1) and (2), we see that a decrease of 1.297 kgs. milk is compensated by an increase of .104 kgs. grain, that is, on an average within this range of variation, 1 added kg. grain is as efficacious as 12.47 added kgs. milk. Professor Warming considers that the experimental accuracy of the data is not so great as to forbid an adjustment designed to yield round figures; he accordingly presents the adjusted figures of columns (3) and (4), thus obtaining, for instance, the figure 6 instead of the 12.47 obtained above. The figures for this and for the three other transitions will be found in column (5). Having thus obtained the ratios of the marginal products, we now turn to consider the marginal products themselves. Here first columns (6) and (7) repeat by proportion (on the above-mentioned

TABLE XI

REPLICATED EXPERIMENTS ON MALTING BARLEY, 1927–1933

(Recorded in the Annual Report of the Rothamsted Station, 1934)

Summary of Average Responses and Interactions, Grain: cwt. per acre

Place	Year	Mean Yield	Average Responses					1st order Interactions				2nd order Interaction	
			N	St. Error for N	P	K	St. Error for P & K	$N \times P$	$N \times K$	$K \times P$	Standard Error	$N \times P \times K$	Standard Error
Rothamsted	1927	*16.6*	+5.2	±1.21	+1.6	—	±1.21	+0.4	—	—	±2.42	—	—
Rothamsted	1928	*16.6*	+3.4	±0.84	+0.8	—	±0.84	+1.1	—	—	±1.68	—	—
Woburn	1928	*18.2*	+1.0	±1.45	−1.2	+2.2	±1.45	−4.1	−6.6	−3.8	±2.90	+7.2	±5.79
Woburn	1929	*28.8*	+1.2	±0.98	−1.4	+2.8	±0.98	−1.6	−6.0	−0.6	±1.96	−3.9	±3.92
Wellingore	1929	*20.2*	+3.3	±0.63	+0.7	+1.2	±0.63	+6.0	+1.4	−0.6	±1.26	+4.5	±2.52
Rothamsted	1929	*22.9*	+3.5	±1.08	+0.7	−0.5	±0.86	+0.8	−1.6	−1.4	±1.71	+4.0	±3.42
Wellingore	1930	*14.7*	+6.1	±0.71	+0.4	+0.5	±0.70	+1.5	+2.7	+1.4	±1.40	+1.6	±2.79
Sparsholt	1930	*13.9*	+1.5	±0.74	+1.4	+0.2	±0.56	−0.3	−1.2	+0.3	±1.12	+0.2	±2.25
Wellingore	1931	*29.8*	−0.3	±0.65	−0.6	+0.3	±0.88	−0.3	−1.0	+1.3	±1.75	−8.2	±3.50
Wye	1931	*22.6*	+3.6	±0.61	+0.9	+0.5	±0.46	+0.3	−1.9	−1.9	±0.92	+3.2	±1.85
Sparsholt	1931	*17.2*	+1.0	±0.62	+0.2	+0.6	±0.43	0.0	−1.0	−0.3	±0.86	+1.0	±1.72
Wellingore	1932	*30.1*	+1.8	±1.18	−1.5	0.0	±0.98	+2.3	+0.5	+0.3	±1.96	+2.0	±3.92
Wye	1932	*28.7*	+2.8	±1.64	+1.4	−3.0	±1.34	−2.6	+0.6	+0.2	±2.69	+3.5	±5.38
Sparsholt	1932	*24.9*	+3.8	±1.64	+0.6	−0.4	±1.38	−0.4	−0.9	+1.1	±2.77	+2.4	±5.54
Wellingore	1933	*23.6*	+3.2	±0.76	+1.8	—	±0.76	+0.5	—	—	±1.53	—	—
Wye	1933	*26.4*	+5.0	±1.46	+2.2	—	±1.46	+1.1	—	—	±2.92	—	—
Weighted Mean*		—	—	—	+0.64	+0.25	—	+0.28	−0.78	−0.29	—	+1.21	—
		—	—	—	±0.21	±0.22	—	±0.42	±0.45	±0.45	—	±0.89	—
Unweighted Mean*		*23.16*	+2.91	—	+0.68	−0.20	—	+0.26	−0.42	+0.11	—	+1.08	—

* 1930–33 and Rothamsted 1929.

The responses to nitrogen are either those to sulphate of ammonia or the mean responses to sulphate of ammonia and nitrate of soda.

The dressings per acre in cwt. were as follows:

1927–28 and Wellingore 1929: 0.2N, 0.486 P_2O_5, 0.75 K_2O.

1930–33 and Rothamsted and Woburn 1929: 0.2N, 0.4 P_2O_5, 0.6 K_2O.

Table XII

Response of Potatoes to N, P, K: Experiment Conducted under the Direction of the Rothamsted Station

By G. Major, Esq., Newton Farm, Tydd, Wisbech, 1933

Plan and Yields in lb. of Individual Plots

$N_0P_0K_0$ 408	$N_0P_2K_1$ 479	$N_2P_0K_2$ 491	$N_1P_2K_1$ 530	$N_1P_1K_2$ 514	$N_0P_0K_2$ 459	$N_2P_2K_1$ 552	$N_1P_1K_1$ 476	$N_0P_0K_1$ 444
$N_1P_1K_0$ 498	$N_2P_1K_1$ 534	$N_1P_0K_1$ 466	$N_2P_1K_0$ 533	$N_0P_2K_0$ 491	$N_2P_0K_1$ 481	$N_2P_1K_2$ 531	$N_0P_2K_2$ 479	$N_2P_0K_0$ 485
$N_2P_2K_0$ 508	$N_0P_1K_2$ 468	$N_1P_2K_2$ 553	$N_1P_0K_0$ 467	$N_2P_2K_2$ 644	$N_0P_1K_1$ 441	$N_1P_0K_2$ 473	$N_0P_1K_0$ 448	$N_1P_2K_0$ 486

Summary: tons per acre
Mean of all Potash (± 0.208)

Superphosphate	Sulphate of Ammonia: None	0.4 cwt. N	0.8 cwt. N	*Mean* (± 0.120)	*Increase* ($\pm$ 0.170)
None	11.70	12.55	13.01	*12.42*	
0.7 cwt. P_2O_5	12.12	13.28	14.27	*13.22*	*+0.80*
1.4 cwt. P_2O_5	12.94	14.01	15.21	*14.05*	*+0.83*
Mean (± 0.120)	*12.25*	*13.28*	*14.16*	*13.23*	
Increase (± 0.170)		*+1.03*	*+0.88*		

Mean of all Superphosphate (± 0.208)

Sulphate of potash	Sulphate of Ammonia: None	0.4 cwt. N	0.8 cwt. N	*Mean* (± 0.120)	*Increase* (± 0.170)
None	12.03	12.96	13.62	*12.87*	
1.0 cwt. K_2O	12.18	13.14	13.99	*13.10*	*+0.23*
2.0 cwt. K_2O	12.55	13.75	14.87	*13.72*	*+0.62*
Mean (± 0.120)	*12.25*	*13.28*	*14.16*	*13.23*	
Incr. (± 0.170)		*+1.03*	*+0.88*		

Mean of all Nitrogen (± 0.208)

Superphosphate	Sulphate of Potash: None	1.0 cwt. K_2O	2.0 cwt. K_2O	*Mean* (± 0.120)	*Increase* (± 0.170)
None	12.14	12.42	12.70	*12.42*	
0.7 cwt. P_2O_5	13.20	12.96	13.51	*13.22*	*+0.80*
1.4 cwt. P_2O_5	13.26	13.94	14.96	*14.05*	*+0.83*
Mean (± 0.120)	*12.87*	*13.10*	*13.72*	*13.23*	
Incr. (± 0.170)		*+0.23*	*+0.62*		

3 randomized blocks of 9 plots each (No replication). Two degrees of freedom for second order interactions are confounded with blocks and the error is estimated from interactions of deviations from regression effects. Plots: 1/60 acre.

Treatments: Sulphate of ammonia at the rate of 0, 0.4 and 0.8 cwt. N, superphosphate at the rate of 0, 0.7 and 1.4 cwt. P_2O_5 and sulphate of potash at the rate of 0, 1.0 and 2.0 cwt. K_2O per acre in all combinations.

Basal manuring: Nil.

Soil: Deep silt, rather heavy. Variety: King Edward. Manures applied: April 17th. Potatoes planted: April 21st. Lifted: September 1st. Previous crop: Peas.

Standard error per plot: ± 0.360 tons per acre or ± 2.7%.

Conclusions.—Significant responses to all three nutrients, with no significant falling off in the responses with the higher dressings. There is a significantly higher response to sulphate of ammonia and superphosphate in the presence of one another, and also to superphosphate and sulphate of potash in the presence of one another. The second order interaction is also significant. The errors are very low, but not exceptionally so for this farm.

assumption) the figures of (3) and (4); thus, in row A we had 1 kg. increase of weight given by 5.8 kg. milk and 2.8 kg. grain, and this is now written in the form (100,000 ÷ 5.8) = 17,200 kg. increase of weight is given by 100,000 kg. milk and 2.8 (100,000 ÷ 5.8) = 48,300 kg. grain. Comparing the successive rows of columns (6) and (7), we see, columns (8) and (9), what increase of weight responds to the increase of grain in each transition. Dividing (9) by (8) and multiplying by 100, we obtain the figures of column (10); and dividing these in turn by (5), we obtain those of (11).[20] Thus finally in (10) and (11) we have the marginal products of milk and grain associated with the different proportionate combinations of diet set out in (6).

(2) The modern technique of field experiment may be illustrated by two examples drawn from the wide range of inquiry conducted by the Rothamsted Station. Each of these is reproduced as a whole from the *Annual Report* of the Rothamsted Station for 1934. In Table XI will be found a summary of experiments carried out on barley, 1927–33; the table is noteworthy for the light thrown on the difficuties of experiment by the wide dispersion of the results recorded, and for the presence of statistically significant negative interactions.

(3) In Table XII are shown in detail the plan and results of one experiment on the relation between inputs of fertilizer and the yield of potatoes. It is interesting here to observe that there is no significant evidence of the operation of diminishing returns within the range of input studied.

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[20] It is evident that column (11) could also have been computed by a process similar to that which gave (10), save that we should now put 100,000 kg. grain, and not milk, as the figure common to all the rows. It is this process applied to row E that gives the figure 16 at the foot of column (11), and thereby, applying (5), of (10).

THE STATISTICAL LAW OF DEMAND FOR A PRODUCER'S GOOD AS ILLUSTRATED BY THE DEMAND FOR STEEL

By Roswell H. Whitman

Although H. L. Moore attempted a statistical analysis of the demand for producers' goods[1] very shortly after he had studied the demand for agricultural commodities with some success, up to the present no satisfactory work in the former field has been done. The modicum of success which has attended the work of the agricultural economists has not been duplicated in the field of producers' goods because of the greater complexity of the latter field and the lack of adequate theoretical and statistical technique for dealing with it.

The present article presents the procedure and results of a statistical study of the demand for steel, which is an important member of the group of "producers' goods." In the first section, I shall state certain hypotheses for the demand function for steel in the form of equations, and then briefly explain their theoretical implications. In the following sections, I shall describe the fitting of the equations to the data and discuss the results obtained.

1. THE THEORETICAL FORMULATION

There are different hypotheses which might be used to explain the demand behavior of steel. The following equations state these hypotheses in a mathematical form:

$$y = ap + b + ct. \tag{1}$$

$$y = ap + b + h\frac{dp}{dt}. \tag{2}$$

$$y(t) = ap(t) + b + \int_{-\infty}^{t} \phi(t-\tau)p(\tau)d\tau. \tag{3}$$

$$\begin{aligned} y(t) &= ap(t) + b + a_1p(t-r) + a_2p(t-2r) \\ &\quad + \cdots + a_kp(t-kr) = ap(t) + b + \textstyle\sum_k a_kp(t-kr). \end{aligned} \tag{4}$$

$$y(t) = ap(t) + b + h\frac{dp}{dt} + \textstyle\sum_k a_kp(t-kr). \tag{5}$$

$$y = ap + b + h\frac{dp}{dt} + cI + dt. \tag{6}$$

The hypothesis expressed by equation (1), where y = quantity demanded, p = price, and t = time, makes the quantity demanded a simple

[1] *Economic Cycles: Their Law and Cause*, New York, 1914.

function of the price of the commodity and time. It implies that the "demand" relation of quantity and price is disturbed only by the trends of the two series. This equation is frequently used in the study of the demand for an agricultural commodity.

The hypotheses, equations (2)–(5), are the result of the work of Evans and Roos. Equation (2), where dp/dt, or p', is the rate of change of price with respect to time, was first advanced by G. C. Evans in 1924.[2] In the equation the constants should have the signs $a<0$, $b>0$, h not determined but probably>0. The term involving p' was introduced "in order to take account of the fact that the demand for a commodity may depend on the rate of change of price (whether, for instance, it is increasing or decreasing) as well as the price itself."[3] Evans believed that in the practical case the sign of the coefficient of the derivative was positive.

This reaction of sales to rising and falling prices had been noted by the classical economists,[4] but they had never attempted to take account of this disturbing factor directly in the demand equation. There is little doubt that the steel market is an example of this type of demand behavior. The supply of steel is never fixed and there can be immediate and pronounced shifts in the quantity-taken. These shifts depend upon the mental state and gambling acuteness of the market. Past prices, and the extent to which current requirements have been made at those prices, are also important. The amount purchased in the present may depend not only on the price at that time, but upon the movement of price at that time.

The second element in the steel market is the movement of prices in the present and the opinion the market holds as to the probable extent and direction of this movement. Rising prices may stimulate purchases, if the rising movement is expected to continue, discourage them if the movement is not expected to be maintained. The briefest analysis leads to the conclusion that consumption will be the result of the strategy of dealers in a shifting market, as well as the result of the present price. Thus, it is fairly certain that the application of equation (2) will be pertinent to the problem.

Not only may the rate of change of price effect demand, but, as already noted, past as well as present prices may influence it. Demand

[2] "The Dynamics of Monopoly," *American Mathematical Monthly*, Vol. 31, No. 2, February 1924, p. 77.

[3] G. C. Evans, *Mathematical Introduction to Economics*, New York, 1930, p. 143.

[4] Alfred Marshall, *Principles of Economics*, 8th Edition, London, 1920, Book III, Chapter IV, p. 112.

F. W. Taussig, "Is the Market Price Determinate?" *Quarterly Journal of Economics*, Vol. 35, 1921, pp. 394–411.

may, in fact, depend upon prices which prevailed indefinitely into the past. Upon this assumption, says C. F. Roos,[5] the equation of demand becomes a Volterra integral equation in the form of equation (3), $p(-\infty)$ is assumed finite, and $\phi(t-\tau)$ is negligible if $t-\tau$ is large and negative. This form of the equation is very satisfactory to the mathematician and can be used for theoretical analysis, but for the purpose of the statistician the approach of Evans is the more convenient. He writes the equation in the form of equation (4), where $a_1, a_2, \cdots, a_k$, form a sequence of constants decreasing in numerical value, and r is a suitable interval of time, say a week or a month; in this way the demand would depend on the price one week, two weeks, etc., before, as well as on the price at that time. We might even make $y(t)$ depend on the price at all times. This equation can, of course, be combined with the one using the rate of change of price term as is done in equation (5).

Roos has suggested a form that is even more general. He says: "It seems incorrect, except in special cases, to write the demand for a commodity as a function of the prices alone of the commodity, or even as a function of the price of several commodities. The demand depends upon the price, the rate of change of price, and upon the accumulation of these effects. In some cases it may even depend upon the rate of production, the acceleration of the rates of production, and upon the cumulation of these effects. The general demand equation must then have this form,

$$G(u_1, u_1', \cdots, u_n, u_n', p, p', t) = \int_0^t p(u_1, u_1', \cdots, u_n, u_n', p, p', t).\text{"}^6$$

This attempt to express demand in as general a form as possible is desirable. In actual investigations, however, the statistician must be satisfied with approximations. Some advance in the generality of demand studies will be obtained if satisfactory use can be made of the limited equation (5).

The last hypothesis is expressed by equation (6) where I is an index of industrial production, and t is time. This hypothesis is somewhat broader than the first and takes account of forces influencing the behavior of the price and production of steel, which an analysis such as that of Moore's had neglected. It is very likely, in fact, that the consumption of steel would vary even though the price did not change. The forces causing this independent variation are usually important elements in the fluctuations known as the business cycle. The demand equation, therefore, should be extended to include these factors, whose fluctuations would otherwise obscure the true relation of quantity and price.

[5] "Dynamical Theory of Economics," *Journal of Political Economy*, Vol. 35, 1929, pp. 632–656.

[6] Roos, *op. cit.*, p. 654.

An index of industrial production is the most convenient summary of the business cycle forces creating fluctuations in the purchases of steel. While other factors may be more important in a causal sense, an equation embodying all the possiblities would be exceedingly complex. It is, therefore, convenient to assume that an index of industrial production is a resultant of these forces and that its movement describes the shift of demand through time.

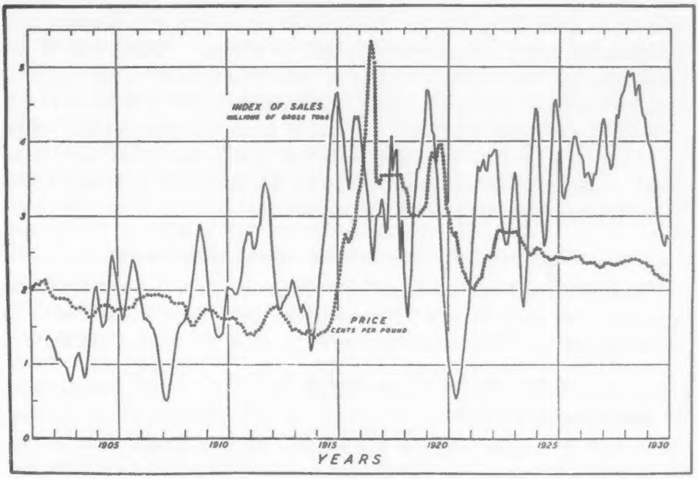

FIGURE 1.—Index of Sales of Steel, and Composite Price of Finished Steel, 1902–1930[7]

It should be noted that time is included as an explicit variable; it acts as a catch-all for the linear trend factors only. Its use is not a denial that the trends in the price of steel may influence the purchases of steel, but an attempt to exclude this problem from that of the cyclical behavior of purchases, which is the field of the present investigation.

The data used for the testing of the hypotheses are an Index of Sales of steel for the period 1902 to 1930, and composite prices for finished steel by the *Iron Age* for the same period. The Index of Sales is a substitute for production figures which are unsatisfactory. The reason for its choice and details of its construction are given in Appendix A. Figure 1 shows the two series for the year 1902–1930.

[7] Source of data: Index of Sales, Appendix A; Price, *Iron Age*, Vol. 127, 1931, p. 133.

For the purposes of fitting the equations, the data were divided into three periods 1902–1915, 1916–1920, 1921–1930. This division was made in order to separate the period of extreme price fluctuation from the more "normal" ones. There is some question whether the use of such periods is in line with the interpretation given their equations by Evans and Roos. Roos thinks that these equations should be applied to a fairly short period in which the movement of prices is in one direction. At the end of an upswing or a downswing, the general solution of equations describing the total economic situation will change, and presumably, the demand equation will be altered.[8] Evans was of the same opinion,[9] but there is in Evans' later statement[10] some justification for the fit to the longer period. He suggested that if crises were not violent, it was possible to have a period of more than one cyclical swing in prices expressed by the same equation. The facts of the case do no great violence to these conditions, and the consistency of the interrelations suggests the selection of periods made.

2. FITTING OF THE SIMPLE DEMAND EQUATION

The first hypothesis expressed by equation (1), in which the constants will have the signs $a<0$, $b>0$, c not determined, did not prove to be an adequate one. The results are as follows: In the period 1902–1915,

$$y = 1.35 - (0.11 \pm 0.41)p + (0.90 \pm 0.15)t;$$

in the period 1916–1920,

$$y = 4.93 - (0.35 \pm 0.16)p - (0.12 \pm 0.06)t;$$

and in the period 1921–1930,

$$y = 2.05 + (0.095 \pm 0.42)p + (1.80 \pm 0.24)t;$$

the sign of p is twice negative and once positive and in only one period is it significant. Such results are too shifting to be interpreted, and, as might be expected from the logic of the hypothesis, it cannot give a satisfactory picture of the demand for steel.

3. FITTING OF THE EVANS-ROOS EQUATIONS.

The Evans-Roos theory of demand is of more use. The testing of it divides itself into three steps. There is the hypothesis in its first form expressed by equation (2), in its second form by equation (4), and in its complete form by equation (5).

[8] C. F. Roos, "A Mathematical Theory of Price Fluctuations," *Journal of Political Economy*, Vol. 38, October 1930, p. 513.

[9] G. C. Evans, Correspondence with Prof. Schultz.

[10] "A Simple Theory of Economic Crises," *Journal of the American Statistical Association*, Vol. 26, March Supplement, p. 67.

The fitting was done as follows: An approximation to the differential term was made by taking the first differences of a price series smoothed by a five-month moving average. In fitting the summation term the effects of remote prices can be disregarded, and the prices of the third, sixth, and ninth months preceding the one in question used.

Inspection of the graph indicates that sales are negatively correlated with past prices. This negative correlation extends to prices considerably remote (more than a year), and in some cases (the third period) sales are apparently more closely related to remote prices than to those of current months. There is reason to believe, however, that this relation is more apparent than real. The graph indicates that the Index of Sales leads prices by approximately ten months, or a quarter period, the whole period being the length of the cycle, about forty months. This means that ten months later prices will be in the same phase of cyclical movement as sales are in any current month and in exactly opposite phase ten months previous, with the result that sales must be negatively correlated with prices ten months previous. In other words, the observed correlations may be the result of the particular timing of the two series in question. As a result of this analysis, it seems unsafe to include prices further in the past than nine months.

Moreover, since the use of prices in adjacent months would make the coefficients so unstable that no satisfactory conclusions could be drawn, only three prices of the nine possible ones were selected. In the second period, the price of the nine months previous was not used because the period was shorter and the correlation between sales and past prices fell off rapidly. The equations were fitted to the trend deviates of the data.

There is one more problem concerning the fitting of these equations. Equation (5) includes both the differential term and the summation term. In a sense, however, both terms measure the same phenomena, and if the price in the month immediately preceding was used, there would be no reason for the use of the rate of change term. However, since the first price in the past used was that of three months previous, it can be argued that using the two terms causes no serious difficulties. It will be seen, subsequently, that equation (5) is the only form of the Evans-Roos equations giving a reasonable fit. Table I gives the results of fitting the three equations. The outstanding points to note are listed below.

In equation (2):

1. The differential or rate of change term, as measured by the size of the b term, is the most important in the estimation of sales; and

2. The sign of the coefficients of price is negative in two cases and positive in one.

TABLE I.—THE INDEX OF SALES[a] FOR STEEL AS A FUNCTION OF PRICES,[b] THE RATE OF CHANGE OF PRICE,[c] AND PREVIOUS PRICES, BY PERIODS, 1902–1930

Type of Equation and Period	a	b	h	a_1	a_2	a_3	R'[d]	ϵ[e]
$y(t) = ap(t) + b + hp'$ Eq. (2)								
1902–1915	-1.00 ± 0.32[f]	3.64	3.19 ± 0.28				0.66	0.47
1916–1920	-0.38 ± 0.12	4.61	0.63 ± 0.11				0.65	0.62
1921–1930	0.60 ± 0.29	0.90	3.15 ± 0.26				0.74	0.57
$y(t) = ap(t) + b + a_1p(t-3) + a_2p(t-6) + a_3p(t-9)$ Eq. (4)								
1902–1915	2.66 ± 0.56	5.81		-3.80 ± 0.91	0.41 ± 0.95	-1.41 ± 0.63	0.58	0.51
1916–1920	0.09 ± 0.19	5.45		-0.48 ± 0.23	-0.24 ± 0.17		0.56	0.68
1921–1930	2.97 ± 0.68	2.94		-2.22 ± 0.80	-0.49 ± 0.67	-0.46 ± 0.43	0.68	0.62
$y(t) = ap(t) + b + hp' + a_1p(t-3) + a_2p(t-6) + a_3p(t-9)$ Eq. (5)								
1902–1915	-2.50 ± 1.00	4.58	3.78 ± 0.64	2.06 ± 1.28	0.02 ± 0.86	-1.07 ± 0.57	0.68	0.45
1916–1920	-0.60 ± 0.25	4.56	0.78 ± 0.21	0.49 ± 0.33	-0.26 ± 0.15		0.67	0.62
1921–1930	-1.93 ± 1.09	-0.68	5.39 ± 0.99	3.57 ± 1.28	-1.26 ± 0.61	0.83 ± 0.46	0.75	0.56

[a] y = sales in millions of tons. (Data for first and last periods corrected for trend.)

[b] p = price in cents per pound. (Data for first and last periods corrected for trend.)

[c] p' = rate of change of price. (First differences of smoothed price quotations.)

[d] R' is the coefficient of multiple correlation corrected for the number of parameters.

[e] ϵ is the standard error of estimate corrected for the number of parameters.

[f] Standard error of the parameter.

In equation (4):

1. Though some of the coefficients are fairly stable, the fit is not as good as in the first case;

2. The signs of a_1, a_2, a_3 are negative and generally decreasing; and

3. The sign of a is positive.

In equation (5):

1. The sign of the coefficient of p' is always positive, and this variable is usually the most important in the estimation of sales;

2. The coefficients of the terms representing past prices are of different signs, but that of $p(t-3)$, i.e., a_1, is always positive; and

3. The coefficient of $p(t)$, or a, is always negative and significant.

The size of the standard errors is due to the large correlations between the various prices. Many of the terms are not significant but will be tentatively retained to aid in the interpretation.

The results of the Evans-Roos hypotheses are difficult to interpret due to the variation in the coefficients obtained by using different forms of the equations. The interpretation is further limited by the fact that we do not have a sufficiently precise a priori hypothesis for the *direction* in which past prices act. It is clear that past prices should have some effect on present sales, but we are not told what the signs of a_1, a_2, a_3, should be, *or what changes in them the introduction of p' should make*. It is certain, furthermore, that the coefficient of p should be negative ($a<0$), and it is probable that the coefficient of p' should be positive ($h>0$). It is well to know this, but knowledge of it is not sufficient to prevent a considerable number of explanations from fitting the facts equally well. I shall state one and make it my tentative conclusion regarding the Evans-Roos theory.

The most simple and important generalization to be drawn from the results of the fitting is that the determination of demand *is due largely to the differential term*. There can be no doubt that the movement of prices definitely influences sales. Comprehension of this fact helps in the interpretation of the remaining coefficients.

The requirement of $a<0$ shows equation (2), with the differential term only, to be inadequate. As a result of fitting this equation the sign of a was twice negative and once positive. It seems difficult to find meaning in either a positive a or its instability of sign.

Careful inspection of the relative importance of the coefficients obtained in fitting equations (4) and (5) will assist the interpretation of these variations.

The difference in the relation of the coefficients of $p(t)$ and $p(t-3)$, i.e., a and a_1, in the two cases is striking. In the first instance that of $p(t)$ is positive and that of $p(t-3)$ negative, while in the second the opposite is true. The change must be due to the introduction of p'

in the second case, but why should such a difference arise? If a positive weight is given the price in the current month, and a negative weight to that of the three months previous, it signifies that rising prices are weighted positively and falling prices negatively. If the prices are about the same the weights will cancel out. A positive coefficient of a differential term, however, amounts to the same thing, for in the latter case also rising prices are weighted positively and falling prices negatively. In other words, in the first case, the weighting of the two prices took the place of the differential term. In the second case the weighting of the present price and of that three months previous may have economic meaning.

It appears from the fitted equation that when the bullish and bearish effects of rising and falling prices are held constant, a price lower than the price three months previous stimulates sales, while a higher price dampens them. This would be partly the result of purchases withheld during the period of falling prices entering the market. There is a possible interpretation, then, that the market rises and falls with rising and falling prices, but that when prices *have* fallen sales will be stimulated, and when they *have* risen sales will be depressed. It is, therefore, possible to conclude that if constant weights are given the rate of change of price and previous prices, the lower the current price the greater will be the quantity-taken, and vice versa.

This hypothesis, however, is not completely satisfactory. Not only is it incomplete, as we have already noted, but the statistical results are not good, taken in their totality.

4. THE FITTING OF THE GENERAL BUSINESS EQUATION

It is probable that the differential term in the Evans-Roos equations is given an importance which is more properly attributed to another variable, namely, *the changes in the general business situation.* It may therefore be better to try an equation in the form of equation (6), where I is the index of industrial production and t is time. For the index of industrial production, the American Telephone and Telegraph Index of Business is used in the first two periods, the Standard Statistics Index of Industrial Production in the last. The equation is fitted to the actual data. In the A. T. & T. index the trend has been removed, while the Standard Statistics index includes the trend. The results of the fitting appear in Table II.

In summarizing the results, the most important relationships established are:

1. The estimates of the Index of Sales from this equation are fairly good.

2. The relations described by the coefficients of regression are uni-

TABLE II.—ANALYSIS OF THE DEMAND FOR STEEL, 1902–1930, BY THE GENERAL BUSINESS EQUATION[a]

(The figures preceded by ± are standard errors.)

Period	$y=b+ap+hdp/dt+cI+dt$	R'[b]	ϵ[c]	${}_{yp}r_{dp/dt\cdot I \cdot t}$[d]
1902–1915	$y=4.20-(1.56\ \pm0.31\ )p+(7.99\pm1.06)dp/dt$ $+(0.036\pm0.005)I+(0.41\pm0.12)t$	0.81	0.45	−0.51
1916–1920	$y=3.84-(0.55\ \pm0.08\ )p+(0.48\pm0.21)dp/dt$ $+(0.12\ \pm0.011)I+(0.15+0.04)t$	0.88	0.39	−0.68
1921–1930	$y=1.49-(1.27\ \pm0.23\ )p+(6.27\pm0.94)dp/dt$ $+(4.64\ \pm0.35\ )I-(0.03\pm0.17)t$	0.92	0.41	−0.53

[a] y = Index of Sales of Steel, millions of gross tons.

p = Price of Steel cents per pound.

dp/dt = First differences of Price.

I = Index of Industrial Production. 1902–1920: A. T. & T. Index, plus and minus deviations. 1921–1930: Standard Statistics Index of Industrial Production—ratios to normal.

t = Time (0.01 is monthly interval), origin at first month in period.

[b] R' is the coefficient of multiple correlation corrected for the number of parameters.

[c] ϵ is the standard error of estimate corrected for the number of parameters.

[d] ${}_{yp}r_{dp/dt\cdot I_t}$ is the coefficient of part correlation. See M. J. B. Ezekiel, *Methods of Correlation Analysis*, New York, 1930, pp. 181–183. Defined generally, it is $r_{x_1x_2.345}$, where $x_{2.345}=x_2-e_{2.345}$.

form and significant and indicate no change in direction from period to period. The net relation between quantity-taken and price is negative, between quantity-taken and the rate of change of price positive, between quantity-taken and an index of industrial production positive, and between demand and time generally positive. There is some change in the order of the coefficients for the war period, but the use of the logarithms of price would probably have reduced this difference. The coefficients of the indices of production are, unfortunately, not comparable. There is some indication, however, that the relations of price and the rate of change of price in the third period returned to their pre-war status.

3. The negative relation between quantity-taken and price can be described in other ways than by the regression coefficient. The part correlation coefficient[11] is, for our purpose, one of the best of these. This coefficient is defined as the correlation between one of the independent variables and the residuals of the dependent variable from its estimate by the other independent variables when these independent variables are given the weights of the general estimating equation. If, for example,

$$X_1 = a + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4,$$

the part correlation coefficient between X_2 and X_1 is the correlation between X_2 and $X_1 - b_{13.24}X_3 - b_{14.23}X_4$. In the present case this coefficient describes the relation between price and quantity-taken after the latter has been corrected, in the manner just described for the three factors, rate of change of price, industrial production, and time. An examination of Table II will show that in all cases this statistic $({}_{12}r_{346})$ is negative and fairly high. It might be argued that while prices should not be corrected for industrial production or for the rate of change of price, they should be corrected for time. Experimenting with this modification in the first period gives a coefficient of -0.51, which is nearly identical with the part correlation. No matter how it is computed, then, a significant net relation of a negative sort exists between quantity-taken and price.

5. COMPARISON OF THE GENERAL BUSINESS EQUATION AND THE EVANS-ROOS THEORY

The general business equation (6) gives, as might be expected, a better fit than the Evans-Roos equation (5). The coefficients of correlation are:

	Equation (5)	Equation (6)
1902–1915	0.68	0.81
1916–1920	0.67	0.88
1921–1930	0.75	0.92

[11] M. J. B. Ezekiel, *Methods of Correlation Analysis*, New York, 1930, pp. 181–183.

The results of fitting the two equations give interesting relationships. In each equation the relation between sales and the rate of change of price is positive and significant. Either demand pattern substantiates the bullish and bearish effect of rising and falling prices. But the outstanding feature is *the stability of the coefficients of the price term* in *both cases.* A comparison of the coefficients of price for the three periods indicate the following.

	Equation (5)	Equation (6)
1902–1915	-2.50 ± 1.00	-1.56 ± 0.31
1916–1920	-0.60 ± 0.25	-0.55 ± 0.08
1921–1930	-1.93 ± 1.09	-1.27 ± 0.23

In none of the three periods is the difference between the coefficients greater than the larger of the standard errors. This stability, of course, does not prove that quantity and price would have a relation of this order under all possible demand hypotheses, but it does indicate that relations obtained by fitting the two equations are more than a chance phenomenon.

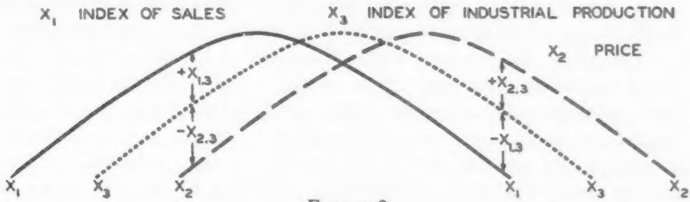

FIGURE 2

Thus, we may say that there now exist by the criteria of rationality, goodness of fit, reasonableness of sign, and size of the parameters, two fairly good demand equations for steel. It is possible to conclude that if either of two different sets of factors is given constant weights, the relation between price and quantity is negative and for both sets approximately of the same order.

The hypotheses made, however, are by no means the only possible ones. For second approximations, equations more complex and flexible might be needed, and when these more refined hypotheses have been stated and tested, the conclusion just drawn may need modification. This conclusion must also be qualified by the fact that the application of a multiple correlation technique gives only an average result for the period covered. Consider, for example, the relations of the cyclical movements of the series in the second experiment. The Index of Sales precedes price by approximately a quarter period. The Index of Industrial Production lags behind the Index of Sales by four or five months, and so will lie between the other two series. Figure 2 which is

an idealized picture of the typical situation, indicates that when the deviations of sales from the Index of Industrial Production are positive ($+x_{1.3}$), those of price are negative ($-x_{2.3}$), and vice versa. But partial correlation can be defined as $r_{12.3}=r_{x_{1.3}x_{2.3}}$, or as the correlation of the residuals of the first and second variables, each from their best estimate, which estimate is calculated from the remaining independent variables. The partial correlation between sales and price will, therefore, be negative, and the sign of the regression coefficient ($b_{12.3}$) negative also. This proves that the timing of the series is responsible, at least in part, for the negative price-quantity relation.

This, in turn, means that the general significance of the coefficients is limited. There is only one observation for each month, only one set of observations for each cyclical movement. The observations for each cyclical movement, furthermore, follow much the same pattern. This limited pattern of observations limits the generality of the coefficients we derive by fitting equations to these observations. Thus, there is no proof that the relations given by the coefficients may not be irreversible. That is, it is not certain, for example, whether a rise of prices early in the boom would have the same effect as one late in the boom. Such a contingency did not arise within the range of the observations. The fitted equations are a description of the average, though fairly uniform, relation of the series. It is quite possible that the action of other unanalyzed factors produces the observed concomitances. An attempt to analyze the possible effect of the position of the cyclical movement on the reaction of quantity-taken to price and to the movement of price will be a pertinent subject for further research.

Despite this qualification, this study is perhaps the first successful attempt to derive from the concrete facts of price behavior of this type of commodity an analysis which is at all plausible. It will, I believe, be the departure for a more rigorous and comprehensive analysis of quantity-price relations.

APPENDIX A

CONSTRUCTION OF INDEX OF SALES

The use of production data for demand study in the steel industry is unsatisfactory, because it involves the assumption that production of one month is approximately equal to the sales of that month. The fact is that the current production of steel products is not sold at current prices, for production is on order and prices are contract prices. There are, however, no sales data available. An index which will approximate sales must be constructed. The series used for this purpose are the monthly figures for the production of steel ingots,[12] and the monthly figures (quarterly prior to 1910), for the unfilled orders of the United

[12] *Iron Age*, Vol. 127, 1930, p. 145.

States Steel Corporation.[13] The construction of the Index was as follows: taking the difference between the unfilled orders of the United States Steel Corporation at the end of successive months gave the monthly change in orders for that company. To estimate the change in orders for the entire industry, it was assumed that the orders of the United States Steel Corporation were in the same proportion to the orders of the entire industry as their production in the current year. If they produced 50% of the steel and had unfilled orders of 4 million tons, the entire industry was expected to have had orders of 8 million tons. If the total increase in the orders of the United States Steel Corporation (calculated) was 0.1 million tons, then that for the entire industry (estimated) was 0.2 million tons. The same results will be obtained if the changes in the unfilled orders of the United States Steel Corporation are weighted by the reciprocal of the proportion of the steel production enjoyed by that company during the current year. The figures for the total steel ingot production of that month were added to these weighted first differences. I have called this series the Index of Sales for steel. Its five-month moving average is plotted in Figure 1.

This index is an approximation to sales from the difference between the orders at the beginning and at the end of the month, plus the amount of the orders that have been filled through current production, and is an estimate of sales. The assumptions we have made are:

1. That the orders of the United States Steel Corporation are representative of the general situation in the industry. The proportion of the production by that company varies from 40 per cent to 60 per cent which precludes the possibility of greatly exaggerating any error in this assumption. The evidence for the assumption is chiefly of a negative sort. The one instance in which a marked difference between the Steel Corporation and the independents was recorded was in 1920, at which time the United States Steel Corporation had booked ahead of the other firms. There resulted for that year an unparalleled price situation in that the prices of the steel corporation were consistently lower than those of the independents.[14] But this is not a frequent occurrence.

2. That there have been no marked changes in the manner in which the United States Steel Corporation records its orders. There is no certain information as to the validity of this assumption. It has been hinted that in recent years the company considers orders that will be filled over a period of time as unfilled orders only at the beginning of the month during which they will be filled. (This is the policy with respect to such orders as those of the American Can Company.)[15] This might be the explanation of the reduction in the lead of sales over pro-

[13] *Ibid.*, p. 148.
[14] *Iron Age*, Vol. 107, 1921, p. 1.
[15] News item in *Chicago Daily News*, February 9, 1932.

duction in the last few years. In the opinion of the trade,[16] however, there has actually been an increase in hand-to-mouth buying. I conclude, therefore, that the changes in the recording of orders have not been sufficiently drastic to invalidate the Index although no adjustment can be made for this factor.

There is one more assumption that must be made. Granted that we have an index of orders, is it a measure of "quantity-taken"? Is it logically the proper dependent variable for a demand equation? We know that production is not the correct dependent variable and that this assumption must be made. But we should, if possible, check the assumption.

It is very difficult to do this. The United States Steel Corporation has never published a statement concerning the exact composition of their figures for unfilled orders. The following facts, however, are common knowledge of the trade:

1. There may be duplication of orders on the upswing and cancellations on the downswing, consequently the Index may exaggerate the cyclical swings in orders.

2. Orders may be placed with no definite specifications for their filling given and such specifications may be delayed indefinitely. A change in orders may, therefore, not represent purchases at the time of placement, but merely indicate probable future purchases. Such orders are made as a guarantee against a rise in prices.

3. A portion of the orders are genuine contracts for definite shapes, and represent actual current sales. So much is known, but the relative importance of the items is impossible to determine. A certain element of the Index of Sales, an element of unknown size, is purely speculative.

The real strength of the procedure in the study lies in this: while it is impossible to get a figure giving the "quantity-taken" of a Marshallian demand equation, it is possible to get an estimate of orders. The demand equation has, therefore, been adapted to this situation. In it there is a term expressing the rate of change of price. When the price starts to rise, the speculative type of order comes in to guarantee against future price rises. If there were no such elements in the steel market and orders were strictly "quantity-taken," it would be much less important to adopt a demand equation which takes this into consideration. As it is, there is a speculative element in the Index of Sales, and a speculative element in the demand equation. The results seem to be the best possible adaptation of method to data and facts.

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[16] *Iron Age*, Vol. 119, 1927, p. 1380.

SUR LA LOI DE LA DEMANDE

Par Otomar Pankraz

Dans cette courte note je vais expliquer une théorie de la demande qui complète—à mon opinion—la théorie de M. Ch. F. Roos. La première partie contient une généralisation de la loi de Cournot et la seconde partie une loi héréditaire de la demande.

I

Si $N = N(t)$ indique un nombre d'unités du bien qui dans un certain domaine économique [resp. dans un certain ensemble des personnes] ont trouvé du débit dans l'instant t et si $p = p(t)$ est le prix d'une unité de ce bien derivé directement de l'observation, alors les individus qui soient actifs en économie décident de la quantité d'unités achetées non seulement selon la hauteur du niveau de prix $p(t)$ [outre les autres facteurs importants], mais aussi

1. selon le veritable changement du prix qui se produisit pendant une courte période immediatement avant le moment t, et
2. selon le changement du prix supposé dans la courte période future *après* le moment t.[1]

Soit t_0 le moment d'observation et $h > 0$ la longueur de l'intervalle de temps. Supposons que la fonction $p(t)$ a la dérivée de gauche $D^-p(t_0)$ dans le point t_0 et introduisons sur l'intervalle $t_0 \leqq t \leqq t_0 + h/2$ la fonction $\chi = \chi(t)$ de la qualité que dans t_0 existe la dérivée du droit $D^+\chi(t_0)$. La forme de la fonction χ il faut choisir selon la catégorie du bien économique. Que ci-après tout au moins ces deux conditions soient remplies:

$$(\alpha) \qquad p(t_0) = \chi(t_0),$$

$$(\beta) \qquad D^- p(t_0) = D^+\chi(t_0).$$

Si

$$\nabla_{(h/2)} p(t_0) = p(t_0) - p\left(t_0 - \frac{h}{2}\right),$$

$$\Delta_{(h/2)}\chi(t_0) = \chi\left(t_0 + \frac{h}{2}\right) - \chi(t_0),$$

puis la somme

[1] Le symbole p il ne faut pas toujours interpreter comme prix. Pour quelques bien de consommation, p peut être par ex. le revenu moyen des personnes d'une certain groupe.

$$(1) \qquad {}_{\chi}\delta_{(h)}p(t_0) = c_1(t_0)\cdot \nabla_{(h/2)}p(t_0) + c_2(t_0)\cdot\Delta_{(h/2)}\chi(t_0),$$
$$[c_1(t),\ c_2(t) \cdots \text{les fonctions données}]$$

définissons comme la différence finie centrale généralisée de la fonction $p(t)$ dans le point t_0.

La différence (1) saisit l'extrapolation du niveau de prix de l'instant passé dans l'instant futur, ainsi que dans le cas général la loi de Cournot a la forme

$$(2) \qquad N = F[p(t),\ {}_{\chi}\delta_{(h)}p(t),\ t],$$

où F soit la fonction essentiellement dépendante de la sorte de la marchandise.[2]

La dépendance de N de la différence ${}_{\chi}\delta_{(h)}p(t)$ on peut expliquer comme l'influence de la spéculation qui provient d'une structure intérieure, autonome d'économie. Si nous appelons selon M. E. Wagemann[3] une telle spéculation comme endogène, puis (2) exprime la loi de la demande avec la spéculation endogène.

De (1) résultent des conséquences très importantes dont quelques-unes furent déjà statistiquement approuvées. Important est le cas ou nous choisissons

$$\chi(t) = p(t_0) + (t - t_0)\cdot D^{-}p(t_0),$$

ainsi que

$$\Delta_{(h/2)}\chi(t_0) = \frac{h}{2}\cdot D^{-}p(t_0).$$

De cette manière nous pouvons très simplement introduire la vitesse d'un certain phénomène économique.

Un second cas important est

$$\chi(t) = a_0 + a_1(t - t_0) + a_2(t - t_0)^2,$$

où on détermine a_0, a_1, a_2, des conditions (α), (β), et de la condition

$$\Delta_{(h/2)}\chi(t_0) = k(t_0)\cdot \nabla_{(h/2)}p(t_0);$$

$k = k(t_0)$ est une fonction du moment t_0.

[2] M. Ezekiel, "Statistical Examination of Factors Related to Lamb Prices," *Journal of Political Economy*, April 1927, 233–260.

H. Staehle, "Die Analyse von Nachfragekurven in ihrer Bedeutung für die Konjunkturforschung, *Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung*, Heft 2. 1929. pp. 16, 29, 35.

[3] E. Wagemann, *Konjunkturlehre. Eine Grundlegung zur Lehre vom Rhythmus der Wirtschaft*, Berlin, 1928, pp. 11–12, 185–186.

E. Wagemann, *Einführung in die Konjunkturlehre*, Leipzig, 1929, pp. 52–53.

H. Kuschmann, *Die Untersuchungen des Berliner Instituts für Konjunkturforschung. Darstellung und Kritik*, Jena, 1933, p. 7.

II

La loi généralisée de Cournot (2) est une fonction du moment t. Mais en réalité déterminent la grandeur de la demande aussi tous les prix passés et tous les changements de ces prix commençants [en première approximation] d'un certain moment fixe $t=0$. A cause de la simplicité qu'il soit dans (1) $c_2(t)\equiv 0$ et (2) soit linéaire en p. Ensuite au lieu de (2) il faut considérer la relation

$$\begin{aligned} N(t) &= A(t) + B(t)\cdot p(t) + C(t)\cdot \nabla_{(h/2)}p(t) + \int_0^t K_1(t,\tau)\cdot p(\tau)\cdot d\tau \\ &\quad + \int_0^t K_2(t,\tau)\cdot \nabla_{(h/2)}p(\tau)\cdot d\tau \end{aligned} \tag{3}$$

$$[A,\ B,\ C,\ K_1,\ K_2 \cdots \text{les fonctions connues}^4],$$

ce qui est l'équation intégrale pour la fonction $p(t)$.

Démontrons que l'équation (3) on peut résoudre comme une équation intégrale de Volterra. Nous allons effectuer cette réduction en général pour l'équation

$$g(t) = f(t) + f_1(t)\cdot g(t-h) + \int_0^t K(t,\tau)\cdot \nabla_{(h)}g(\tau)\cdot d\tau \tag{4}$$

$$0 \leqq t \leqq L(L = \text{const. resp.} + \infty).$$

Choisissons les suppositions:

1. $0<h<L$;
2. $g(t)$ est une fonction continue, connue dans $-h\leqq t<0$;
3. $f(t)$, $f_1(t)$ sont les fonctions continues, connues dans $0\leqq t\leqq L$;
4. qu'il soit

$$\lim_{t\to 0-0} g(t) = g(0-0) = g(0)$$

et

$$g(0) = f(0) + f_1(0)\cdot g(-h);$$

5. le noyau $K(t,\ \tau)$ est une fonction continue dans $0\leqq\tau\leqq t\leqq L$ autrement elle est partout $=0$.

La différence finie $\nabla_{(h)}g(t)$ a $0\leqq t\leqq L$ pour l'intervalle de définition et elle y est évidemment continue. Si l'équation (4) a une solution en

[4] Ch. F. Roos, "Theoretical Studies of Demand," ECONOMETRICA. Vol. II, No. 1, 1934, pp. 73–90.

Ch. F. Roos, *Dynamic Economics*, Bloomington, Indiana, 1934, pp. 14–18, 62–64.

général, puis il s'ensuit des suppositions, que cette solution dans $-h \leqq t \leqq L$ est continue.

La solution de l'équation (4) il faut toujours chercher sur l'intervalle $i \cdot h \leqq t < (i+1) \cdot h$ sous la supposition que nous connaissions déjà la solution sur l'intervalle $(i-1) \cdot h \leqq t < i \cdot h$ pour $i = 0, 1, 2, \cdots$. Si nous définissons la fonction continue,

$$F(t) = f(t) + f_1(t) \cdot g(t - h) - \int_0^t K(t, \tau) \cdot g(\tau - h) \cdot d\tau,$$

puis (4) se reduit à l'équation intégrale de Volterra

$$g(t) = F(t) + \int_0^t K(t, \tau) \cdot g(\tau) \cdot d\tau. \tag{5}$$

Les équations (4) et (5) sont équivalentes. Comme (5) a seulement une seule solution continue, alors l'équation (4) a aussi cette qualité.

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ON A MIXED DIFFERENCE AND DIFFERENTIAL EQUATION

By R. W. JAMES and M. H. BELZ

IN A RECENT paper,[1] Frisch and Holme have shown that equations of the form

$$(1) \qquad \dot{y}(t) = ay(t) - cy(t - \theta), \qquad (a, c, \theta \text{ real constants}),$$

yield periodic solutions of the form

$$(2) \qquad y(t) = ke^{vt/\theta} \cos\left(\frac{ut}{\theta} + \phi\right), \qquad (k, \phi \text{ constants}),$$

where $\rho\theta(\equiv v+iu)$ is determined as a possible solution of the equation

$$(3) \qquad \rho = a - ce^{-\rho\theta}.$$

These authors have studied the characteristic solutions of the equation (3) for the case in which a, c, θ, are all positive. In Kalecki's theory,[2] of which the paper of Frisch and Holme is an outcome, equations of the type (1) occur in which the constants are essentially positive, but in other econometric applications the same type of relation is found without this restriction on the values of a, c, θ. It is the purpose of the present brief note to indicate how the analysis of Frisch and Holme can be extended to embrace the general case in which the constants may have any real values. For instance, the constant θ represents a lag, which may equally well be positive or negative.

Multiplying (3) throughout by θ, putting $\rho\theta = v+iu$, and separating the resulting equation into real and imaginary parts, we find

$$(4) \qquad v = a\theta - c\theta e^{-v} \cos u,$$

$$(5) \qquad u = c\theta e^{-v} \sin u.$$

Combining these equations, we get

$$(6) \qquad v = a\theta - u \cot u.$$

Now (5) may be written

$$(7) \qquad 1 = c\theta e^{-v} \frac{\sin u}{u} = |\, c\theta \,| \cdot e^{-v} \cdot \left|\frac{\sin u}{u}\right|,$$

and, therefore, we have

$$(8) \qquad 0 = \log_e |\, c\theta \,| - v + \log_e \left|\frac{\sin u}{u}\right|.$$

[1] Frisch and Holme, ECONOMETRICA, Vol. III, p. 225.
[2] Kalecki, ECONOMETRICA, Vol. III, p. 327.

Eliminating v between (6) and (8), we get

$$a\theta - \log_e |c\theta| = u \cot u + \log_e \left| \frac{\sin u}{u} \right|, \tag{9}$$

or, using the notation of Frisch and Holme,

$$C = f(u), \tag{10}$$

where $C = a\theta - \log_e |c\theta|$ and $f(u) = u \cot u + \log_e |(\sin u)/u|$. The function $f(u)$ is identical with that plotted by Frisch and Holme when u lies in the ranges $(2r\pi, 2r\pi+\pi)$, $(r=0, 1, 2, \cdots)$, and with the function $u \cot u + \log_e \{(-\sin u)/u\}$ when u lies in the ranges $(2r\pi+\pi, 2r\pi+2\pi)$.

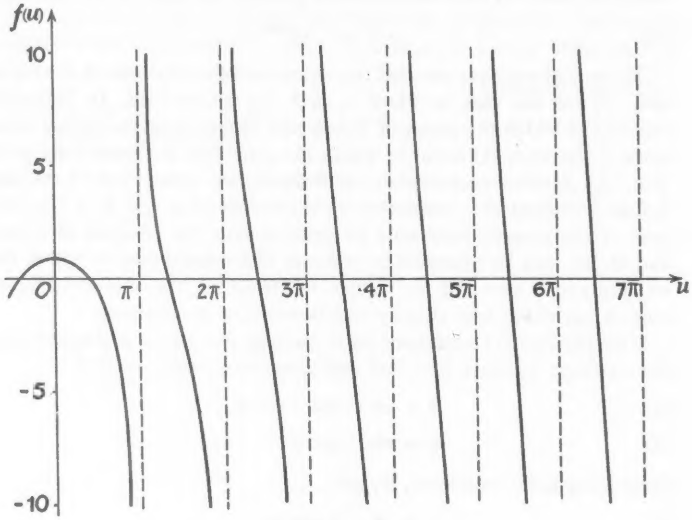

FIGURE 1

The accompanying Figure 1 gives the graph of $f(u)$ when the range of the argument is $(0, 7\pi)$, and consists of the addition of the branches in the intervals $(\pi, 2\pi)$, $(3\pi, 4\pi)$, $(5\pi, 6\pi)$, to those given by Frisch and Holme for the remaining intervals. The data for the special ranges referred to are embodied in another form in the following Table.

The sign of $c\theta$ determines whether admissible solutions of the equation (10) shall lie in a range given by $(2r\pi, 2r\pi+\pi)$ or by $(2r\pi+\pi, 2r\pi+2\pi)$. From (5), we infer that $(c\theta \sin u)/u$ is essentially positive.

Accordingly, when $c\theta$ is positive, that is, when c and θ are of the same sign, $(\sin u)/u$ must also be positive. This restricts u to the ranges given by $(2r\pi, 2r\pi+\pi)$, the case covered by the authors referred to. When c and θ are of opposite signs, $(\sin u)/u$ must be negative, which limits u to the ranges given by $(2r\pi+\pi, 2r\pi+2\pi)$. When the integer r is large and $c\theta$ is negative, an approximate solution of (10) is given by $u=(2r+3/2)\pi$.

TABLE I

u/π	$f(u)$	u/π	$f(u)$	u/π	$f(u)$
1.00	∞	3.00	∞	5.00	∞
1.20	3.3304	3.20	10.9976	5.20	19.160
1.25	2.2125	3.25	7.5190	5.25	13.344
1.30	1.3482	3.30	4.9817	5.30	9.0729
1.32	1.0401	3.35	2.8933	5.35	5.6266
1.34	0.74488	3.37	2.1358	5.40	2.6308
1.35	0.60074	3.39	1.4077	5.41	2.0643
1.37	0.31713	3.40	1.0519	5.42	1.5052
1.39	0.03718	3.42	0.35236	5.43	0.95206
1.40	−0.10231	3.45	−0.67884	5.44	0.40376
1.42	−0.38190	3.47	−1.3628	5.45	−0.14093
1.44	−0.66427	3.48	−1.7059	5.46	−0.68315
1.45	−0.80719	3.49	−2.0506	5.47	−1.2240
1.47	−1.0979	3.50	−2.3975	5.48	−1.7647
1.49	−1.3969	3.51	−2.7476	5.49	−2.3061
1.50	−1.5502	3.55	−4.1905	5.50	−2.8495
1.51	−1.7064	3.60	−6.1506	5.55	−5.6325
1.53	−2.0288	3.65	−8.3975	5.60	−8.6340
1.55	−2.3666	4.00	$-\infty$	5.65	−12.036
1.56	−2.5422			6.00	$-\infty$
1.57	−2.7227				
1.60	−3.2981				
1.65	−4.4021				
2.00	$-\infty$				

If u_1 is a solution of (10), the damping exponent for the corresponding oscillation will be given by

$$v = \log_e |c\theta| + \log_e \left| \frac{\sin u_1}{u_1} \right|. \tag{11}$$

When θ is positive, it is easily shown that the oscillation will be damped, undamped, or explosive, according as $|c|$ is less than, equal to, or greater than, $|u_1/(\theta \sin u_1)|$. When θ is negative, the condition for damping is that $|c|$ shall be greater than $|u_1/(\theta \sin u_1)|$.

In regard to possible real exponential solutions of (1), the case where c, θ, are both positive has been treated previously. It is found that there will be two real roots of (3) if $a>c$ and if $a<c$, two, one, or none, according as C is greater than, equal to, or less than, 1. When $C=1$,

solutions of (1) will be of the form $e^{\rho t}$ and $te^{\rho t}$. On the other hand, when c, θ are both negative, there will be two real roots of (3) if $a<c$ and if $a>c$, two, one, or none, according as C is less than, equal to, or greater than, 1. When c and θ are of opposite signs, one real exponential solution of (1) will always exist. When ρ is real, the solutions of (3) will be obtained from the abscissae of the points of intersection of the graphs of the functions $ce^{-\rho\theta}$ and $a-\rho$. For the case here considered, there will be one such point of intersection for each set of values of a, c, θ. When c is negative and θ positive, ρ will be greater than, equal to, or less than, 0, according as a is greater than, equal to, or less than, $-c$; while if c is positive and θ negative, ρ will be greater than, equal to, or less than, 0, according as a is greater than, equal to, or less than, c.

We desire to thank Mr. J. Walpole, B.Sc., for assistance in the computations.

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RELATIONS OF INSTITUTIONAL FACTORS TO ECONOMIC EQUILIBRIUM AND LONG-TIME TREND

By Elmer C. Bratt

(A) NATURE OF THE LONG-TIME TREND AND NEED FOR ITS FORECASTING

The importance of anticipating future changes in the conduct of business operations has been recognized for some time. The work of such men as Brookmire, Persons, and Babson, is well known. Major emphasis has been placed on forecasting changes for the following few months. But little attention has been given to forecasting long-time movements. Although it is clear that students have recognized the importance of this latter problem, the difficulties involved have generally appeared almost insuperable.[1] It is the purpose of this paper to present a theoretical background for the solution of this problem when applied to the long-time trend of total industry.

The necessity of forecasting long-time trends is fairly obvious upon reflection. Since, in general, production is round-about, it is necessary to produce large quantities of capital equipment. We know but little about the average life of such equipment, but probably it is not less than ten years.[2] After such equipment is built, the uses to which it can ordinarily be put are strictly limited. The building of such equipment implies, therefore, that usufruct can be sold during the life of the capital. Failure to study the need for such usufruct during the entire life of the capital has a close bearing upon economic unbalance.

[1] W. M. Persons said fifteen years ago, "the considerations upon which an estimate of trend is based are identical with those at the basis of all plans for the future." "Indices of Business Conditions," *Review of Economic Statistics* 1: 5–107. Twelve years ago L. P. Ayres suggested that "the estimating of future demand . . . depends largely on the estimating of long-time or secular trends." "The Nature and Status of Business Research," *Journal of the American Statistical Association* 18: 33–41. About the same time W. C. Clark stated that one of the two statistical problems of most immediate importance is "the further analysis of the concept of a 'normal' in industry," *Journal of the American Statistical Association* 18: 1053–56. But Carl Snyder wrote in 1926, "it is thus evident that long-time prediction of trends is unwise and likely to be inaccurate." *Business Cycles and Business Measurements*, 1927, p. 54.

[2] Important data on the life of industrial machinery can be found in R. Winfrey and E. B. Kurtz, "Life Characteristics of Physical Property," *Iowa State College of Agricultural and Mechanical Arts Official Publication*, Bulletin 103; American Machinist, *1930 Inventory of Metal Working Equipment*, 1931; Standard Trade and Securities, General Section, volume 68, number 38, June 28, 1933, "Industry Control and Obsolescence."

The production of capital goods is over-concentrated in prosperity periods. Decision to make capital investments in excess at such a time seems to be largely due to the fact that corporate managements misinterpret capital needs indicated by current sales.[3] It is reasonable to conclude that a more rational method of anticipating capital needs should make for less concentration of capital production in prosperity periods, and hence modulate the business cycle to some extent.

Keeping business cycle measurements up to date ordinarily involves forecasting long-time trends. It is not feasible to remeasure the trend as each new figure becomes available; to do so would require endless revision of the business cycle measurement. As a practical proposition, it is seldom feasible to use for elimination trend-line measurements made with data nearer than a few years back of the present. This is true since trend-line measurements must begin and end in the same phase of the cycle, and it is very difficult to estimate the same phase at other than high and low points until an accepted trend measurement is had.

The definition of the long-time trend occupies a crucial position. It has been typical to define long-time trends empirically. Such definitions range from "the result obtained by any statistical measurement" to Burns' definition "as the 'non-cyclical' movements of series."[4] The definition of the long-time trend as "a path of balance" typically has been considered question-begging. Indeed, the conclusion is assumed if any particular statistical measurement is defined as the path of balance. If the long-time trend is conceived as tracing the path of equilibrium, it will be necessary to dèmonstrate that some particular statistical measurement follows this path. If such a procedure is followed, a measurement can be accepted as locating the position of the long-time trend only after it is demonstrated that such a measurement traces the path of equilibrium.

Defining the long-time trend as the "path of balance" reduces the forecasting of it to manageable proportions. As the long-time trend usually has been defined and analyzed, many students feel its very use "is essentially a confession of ignorance of some of the important factors involved or is the result of a desire to discuss these factors without identifying them."[5] Typically the long-time trend has been measured in order that it might be eliminated, for the purpose of

[3] This problem has been well analyzed by J. M. Clark in his discussion of overhead costs. See, for instance, his article, "Business Acceleration and the Law of Demand: a Technical Factor in Economic Cycles," *Journal of Political Economy* 25: 217–35, March, 1917.

[4] A. F. Burns, *Production Trends in the United States Since 1870,* Nat. Bur. of Econ. Res., 1934, p. 31.

[5] C. F. Roos, *Dynamic Economics,* 1934, p. 4.

studying short-time changes. In such elimination, it is not common to attempt any analysis of the trend eliminated.

Some students have conceived the long-time trend as containing a primary trend with a secondary trend or long cycle varying about it.[6] The general nature of the secondary trend might be a variation in the rate of growth of the primary trend, or it might be conceived as a slow shift about balanced conditions. In the former case there is conceived an alternate speeding up and slowing down in fundamental, organic changes, and in the latter there is conceived a variation about such fundamental, organic changes. In the former case the secondary trend must be included in the measurement of the long-time trend conceived as tracing the path of balanced conditions. The description of the secondary trend given by qualified students, however, implies a variation about balanced conditions.[7] In any case the precise nature of the secondary trend is not known. Probably a bifurcate division of the trend into primary and secondary trends is an over-simplification. Need for wholesale analysis and comparison of trends may justify it, but, if what is desired is the analysis of one trend in aggregate detail for forecasting purposes, a better procedure is to consider the long-time trend of total industry as the path of balance, whether this does or does not include a part or all of what would appear as the secondary trend by cursory analysis. In this method, the important factors of variation about the path of balance, of longer duration than the business cycle, will come up for consideration with the semi-qualitative factors determining the long-time trend. For the long-time trend of total industry, these factors are briefly analyzed in the latter part of this paper.

If we identify the true trend level with the normal level, the trend level at any given time expresses the level of consumption to be expected, on the average, with current institutions, habits of thought, and technological ability. It is the level to be expected from long-run tendencies. This idea is synonomous with the concept of equilibrium in static economics. The actual level of production is highly artificial most of the time. The long-time trend level is the level at which economic balance would be achieved at any time.

[6] The principal studies have been made by Van Gelderen, Kondratieff, Kuznets, and Wardwell. See the summarization and references given in W. C. Mitchell's *Business Cycles*, 1927, pp. 227–30; Simon Kuznets, *Secular Movements in Production and Prices*, 1930; C. A. R. Wardwell, *An Investigation of Economic Data for Major Cycles* (published privately, Philadelphia, 1927).

[7] Kuznets gives first importance to the long cycles in prices in explaining secondary trends. Wardwell attributes major cycles to the over-expansion of fixed assets. See references in footnote 6.

(B) ECONOMIC BALANCE AND EQUILIBRIUM

Practically all economists recognize a study of the conditions of equilibrium as essential to economic analysis. The level of output under conditions of equilibrium is often said to be *normal.* As defined by Haney, "business is said to be 'normal' when, in general, supply and demand are so adjusted that the prices of commodities, on the average, equal cost of production plus the usual competitive profit."[8] It is probable that a more practicable concept would be one involving an equivalence of cost and price at the "bulk line" or margin rather than at the average.[9] Also, there must be some reasonable balance between various prices. It is conceivable that such an average or "bulk-line" price level might be achieved by unwarranted stimulation to one group of industries while prices in other industries were far under cost of production. If normal is defined in terms of price, therefore, the phrase, "assuming a reasonable balance between various prices," should be added. Such a concept of equilibrium involves the idea of price as the regulator. It is possible to conceive of this path of balance directly in terms of quantity. It may be defined as the level where production and consumption are exactly equal when proper allowance is made for depreciation, obsolescence, replacements and repairs of machinery, of equipment, and of durable consumers' goods. Theoretically, this level is equivalent to normal levels defined in terms of price. With price and cost equivalent at the "bulk-line" or margin, such market conditions are implied as that there is taken from the producer the quantity he produces. If prices were below costs, the producer would tend to restrict production, while if prices were above costs, the producer would tend to over-expand production. In actual application, the theoretical notions of price involve many complexities which might make the time of occurrence of the two types of balance differ somewhat. Price and cost might balance in the production processes but not in the distribution processes. Even though costs and prices balance at the margin, there may be a piling up of inventory on balance due to anticipation that prices will go still higher. The general theoretical notion of price is useful in understanding the concept of a balance between production and consumption; in application it is simpler to think in terms of this latter concept. As pointed out later, such a concept is applicable to total industry only, and is not acceptable as the definition of normal for any given industry.

Haney advises against indentifying normal with the statistician's long-time trend line. Such a procedure certainly is incorrect if there

[8] L. H. Haney, *Business Forecasting*, 1931, p. 8.

[9] For a discussion of bulk-line costs, see F. W. Taussig, "Price-Fixing as Seen by a Price-Fixer," *Quarterly Journal of Economics* 33: 205–41, February, 1919.

is involved merely the identification of the characteristics of normal with a random trend-line measurement. An entirely different approach, however, is to define the true long-time trend as normal; and then attempt to measure this normal. Such a procedure does not imply that any measurement traces the path of balance, but requires that the measurement be shown to be the path of balance. This implies an analysis of the factors determining the long-time trend, not a confession of ignorance regarding them.

(c) CHANGING POSITIONS OF EQUILIBRIUM

The shift in the long-time trend over time expresses the fundamental change in economic conditions.[10] This is not a static but a dynamic concept. Changes in balanced levels are properly spoken of as structural changes. Structural changes may be quantitative or qualitative, or, as Wagemann expresses it, may be either continuous or discontinuous. Quantitative changes are principally those due to changes in population number. Qualitative changes result from such factors as the growth of new industries, the decline of old industries, commercial application of new inventions, changes in the legal rules governing the method of doing business, changes in habits and in institutions. Probably it is principally the existence of qualitative or discontinuous changes which leads Wagemann to declare that "previous experience forms a fundamentally inadequate basis in [trend forecasting]."[11] Since there is no quantitative change easily ascertainable from any one of these qualitative changes, it is sometimes implied that the changes in normal levels cannot be measured. What is commonly overlooked is that *quantity* changes in the long-time trend are chiefly the result of these qualitative changes. Later in this paper, we infer how it is possible to make rough quantitative judgments regarding the probable influence of the various qualitative factors determining the trend of total industry.

There are reasons for believing that the changes in the level of the long-time trend of total industry are generally slow, accretional changes. Whether such is most probably true at any given time is best answered in terms of the factors determining the long-time trend. As a general proposition, however, cultural changes, which are dependent upon shifts in habits and institutions, may be said to be "selectively accumulative in time."[12] By slow outcroppings from the

[10] An excellent statement of this is given by O. Gressens, "The Quantitative Determination of Fundamental Changes in Economic Data," *Journal of the American Statistical Association* 20: 549–55, December, 1925.

[11] E. Wagemann, *Economic Rhythm*, 1930, p. 213.

[12] This is the expression used by F. S. Chapin in his *Cultural Change*, 1928.

inertia of the past, items are dropped and items are added. The long-time trend rests not only upon prevalent culture, but also upon materialistic, technological changes. There is a predominant opinion at the present time that any specific invention depends upon a certain cultural preparation and could not be made without the existence of the constituent cultural elements, and that such general societal culture together with scientific preparation and development determine what inventions will be made.[13] Innovations and startling transformations occur, but ordinarily they are of relatively small importance compared to all of the slow changes taking place simultaneously. A revolution may set up ideals which are a complete break with the past, but reaction is certain, and changes in manner of living will be an outgrowth of a long accumulation of small changes.[14]

It is useful to point out the nature of the long-time trend of any given industry. Long-time trend levels of individual industries can be defined as the levels required for each and every industry to produce a normal level for total industry. The normal level of production for any industry is properly conceived as a part of the normal level of total industry. This proportion is a variable as it must be conceived of as changing from time to time. A normal level for industry at any given time will require, however, some particular quantity of goods from each industry. It is not very difficult to demonstrate that the definition of the long-time trend level for total industry cannot be applied to any given industry. Assume, for simplicity, in an individual industry, that no inventories are being accumulated, that production is at a very low level, that the rate of replacing capital equipment is greatly curtailed but yet sufficient to support existing rates of production. There would exist the only equivalence between production and consumption which can occur for an individual industry. All of the final product that is being produced is being consumed; and capital equipment is being maintained for such rates of production. But such a condition is, of course, not a reasonable conception of normal or long-time trend levels. Further, it is possible for an individual industry that, as production declines, prices may be *raised* so much that they actually cover cost of production at unreasonably low levels.

[13] See W. F. Ogburn and D. Thomas, "Are Inventions Inevitable?" *Political Science Quarterly* 37: 83–98, March, 1922; R. C. Epstein, "Industrial Invention: Heroic or Systematic?" *Quarterly Journal of Economics* 40: 232–72, February, 1926; R. K. Merton, "Fluctuation in the Rate of Industrial Invention," *Quarterly Journal of Economics* 49: 454–69, May, 1935.

[14] Ogburn points out, "it is thinkable that the piling up of these cultural lags may reach such a point that they may be changed in a somewhat wholesale fashion." *Cultural Change with Respect to Cultural and Original Nature*, 1922.

(D) RELATION OF PRODUCTION TO CAPACITY TO CONSUME AND CAPACITY TO PRODUCE

If the long-time trend level of total industry is accepted as the normal, fundamentally balanced economic condition which we have described, the upper limiting condition of the long-time trend level must be accepted to be the capacity to produce rather than the capacity to consume. This is true because there is a scarcity of total economic goods.[15] If production automatically created its own consumption, i.e., if "Say's Law of Markets" were literally true, the long-time trend would lie at the level of the capacity to produce. In actual practice, this is not the case. The Brookings Institution studies seem to show that, at the peaks, industry operates at about 80 per cent of full capacity.[16]

The lower limiting condition of the long-time trend at any given time is fixed by the spending habits involved in so-called "irreducible" standards of living. That people develop a way of life which is dependent upon certain minimum requirements in the matter of physical goods seems to be a sound observation. Probably many of our institutions rest upon such ways of living. Certainly a large proportion of our individual habits are dependent upon such a minimum quantity of goods. To decrease such minimum consumption would involve an overcoming of much of the inertia of the past. Such is quite unlikely unless forced by bare necessity.

The goods which must be produced at the lower limiting condition are the "indispensable" goods, except insofar as inventories of these goods are available. Such goods comprise all short-lived or transient goods in the "irreducible" standard plus a minimum quantity of replacements of durable goods required at specific points in order to maintain the "irreducible" standard. All other durable goods and short-lived goods not contained in the "irreducible" standard are "dispensable" goods. This does not mean that durable goods are any less necessary at minimum levels than are short-lived goods, but that durable goods, on the whole, can be used without being currently produced. The extent to which production falls to minimum requirements seems to vary greatly between one low production period and another. There is a great variation in the apparent speculative possibilities involved in producing durable goods at low cost when there is no imme-

[15] This has been indicated empirically by many students by hypothetical allocation of the national income at its peak. See, e.g., M. Leven and Associates, *America's Capacity to Consume*, Brookings Institution, 1934.

[16] E. G. Nourse and Associates, *America's Capacity to Produce*, Brookings Institution, 1934.

diate use for them. Further, the "originating" causes[17] of business cycle variations seem to have very different effects in different depressions.

As already noted, there seems to be a clear tendency for production to reach about the same proportionate distance from capacity at each peak. Further, it is obvious upon examination that the peaks of available indexes of total production tend to reach an approximately uniform distance above a measured long-time trend at the various prosperities.[18] We can conclude, therefore, that the long-time trend tends to bear a reasonably uniform proportion to the capacity to produce, but that it does not tend to bear any very close relation to the minimum requirements of consumption. It cannot be taken for granted that the long-time trend will always bear a constant proportion to capacity but, with careful examination to see that factors are not at work tending to shift the proportion, it will form a reasonable working hypothesis.

The equilibrium between the forces of supply and demand in static economics must be thought of as a practical working balance rather than as a purely theoretical one. If we think of such a balance as one occuring when industry is operating at full capacity, we have an unattainable situation. Industry never operates at full capacity. This is explained by the characteristic movement of business cycles. As noted above, durable goods being, on the whole, "dispensable," are produced at very limited rates during periods of low activity, while their production is over-concentrated in the period of high activity. A part of such durable goods is composed of capital goods. In periods of high activity, therefore, a part of the productive forces is not available for the production of consumers' goods. The fact that it was not actually being used in the production of consumers' goods in the phase of low activity has nothing to do with the point, for consumers' goods were being produced at somewhat restricted levels. When peak levels are attained, such productive forces are no longer available for the production of consumers' goods. The proportion of forces withdrawn from the potential production of consumers' goods is greater than indicated by the average proportion of productive effort diverted to the production of capital goods because the variation in the production of

[17] As used by Clark, such causes are those which originate outside the business system. See *Strategic Factors in Business Cycles*, National Bureau of Economic Research, 1934, pp. 14–21.

[18] The difficulty involved in giving a precise answer to this depends upon the inadequacy of our measures of total production as noted below. Any index, however, will give a rough indication of the facts. Persons' Index of Industrial Production and Trade for the United States, 1900–1930, for instance, never reached a peak less than 5.5 per cent above normal nor more than 13 per cent above normal, if the World War years are eliminated.

capital goods is greater than in the production of consumers' goods.[19] At the peak many of the producers' goods have been completed. At such time the "period of gestation" in the production of capital goods has been completed. It is obvious that, whenever recession sets in, a large supply of capital goods will have just become available. There is no reason to believe that, just because this additional capacity has become available, there will be an immediate increase in the demand for consumer's goods which such additional capacity could produce. Therefore, at the peak we do not operate at full capacity.[20]

In actual practice there can be no such thing as a balance between production and consumption with industry operating at full capacity. It is possible, however, for a balance between production and consumption to be attained at the long-time trend level in the sense that for the goods consumed there is an equal quantity produced while, at the same time, there are produced just enough durable goods to allow for depreciation, obsolescence, replacements, and repairs. At such a level, industry does have an over-capacity. With our type of economic organization, it is not possible to obtain a balance with industry operating at full capacity. The idea of a balance at such levels conforms to the old conception of periodic crises, rather than to that

[19] The typical difference in the violence of fluctuation is probably greater than indicated by Leong's indexes, due to the unavailability of data for many transient goods industries; to the fact that those for which data are available are standardized commodities only; to the fact that these commodities are, in general, more readily stored than those for which data are not available; and to the fact that one unconsciously gives great weight to the similarity of movement in the 1921 depression when there was a very unusual deflation of inventories. See Y. S. Leong, "Indexes of Physical Volume of Producers' Goods, Consumers' Goods, Durable Goods, and Transient Goods," *Journal of the American Statistical Association*, 30: 361–76, June, 1935.

Looked at in another way, there is clear evidence that the violence of fluctuation in the production of durable goods is much greater than in the production of transient goods, but apparently durable consumers' goods fluctuate no more widely than do capital goods. Probably a larger proportion of consumers' goods is transient than is true in the case of producers' goods.

[20] H. G. Moulton, in *The Formation of Capital*, Brookings Institution, 1935, is greatly impressed by the fact that there is a "persistent failure to make full use of our productive resources" and, at the same time, "a chronic state of under-consumption on the part of the great masses of the people" (p. 4). Unequal distribution of income is implied as the cause of this situation. Moulton recognizes that "the search for some *single cause* of business fluctuations has, on the whole, been as confusing as it has been fruitless." It appears, however, that he has fallen into this trap. For, on the one hand, our analysis indicates that over-capacity, even at the peaks, is a characteristic of the business cycle. On the other hand, under-consumption on the part of the masses, in the sense they would gladly consume more goods below the level of "desire for superiority" (in the Keynes' sense), is a characteristic of a scarcity economy.

of business cycle variations. With only periodic crises, business might normally balance while operating at full capacity; during the period of a crisis, it would temporarily fall from such levels. The fact is, however, that operation at peak levels is as unbalanced as at low depression levels. At peak levels durable goods are being created at a rate considerably in excess of the rate of current use of their usufruct. Balance of long-run forces occurs at long-time trend levels. It is true, although lamentable, that this balance does not occur at the peak of mechanical capacity.

In this section we have stated four general propositions. The upper limiting condition of the long-time trend is the capacity to produce. The lower limiting condition of the long-time trend is the necessity to consume. The long-time trend tends to bear a fairly constant proportion to the first, but not to the second, of these conditions. As a practical matter, balance cannot be achieved at the mechanical capacity to produce.

(E) ECONOMIC GROWTH

The attempt to develop functional laws of economic growth has a long history. Stanley Jevons developed the "natural law of social growth."[21] This law is merely a compound interest curve. Jevons conceived this constant relative rate as applying to population and to industries generally, although he recognized that industries cannot go on doubling forever, that an industry's growth has "a natural limit of convenience, or commercial practicability" beyond which growth cannot well proceed. To Jevons, this seemed like a paradoxical situation which would lead to disaster without artificial intervention.

Partially as a result of the work of Malthus, the general opinion during the nineteenth century was that population grows according to a constant relative rate. To many students, it became obvious by the close of the century that population was not so increasing. During the nineteen-twenties, Raymond Pearl and L. J. Reed popularized the logistic curve as a method for depicting the growth of population, fitting it to all available information on population growth as well as to the growth of many biological characteristics.[22] The symmetrical logistic describes a curve whose relative rate of growth declines throughout its range from a lower to an upper asymptote. From the apparent success of Pearl and Reed in applying the logistic curve to population, there grew up a wide claim that the logistic curve (or the Gompertz, which is a curve having somewhat similar characteristics)

[21] See W. Stanley Jevons, *The Coal Question, An Inquiry Concerning the Progress of the Nation, and the Probable Exhaustion of Our Coal Mines*, second edition, revised, 1866.

[22] See particularly Raymond Pearl, *The Biology of Population Growth*, 1925.

states the general law of growth applicable to many individual industries.[23]

Although this type of curve often has been suggested as the description of total industrial growth, no reasonably good measurement of this long-time trend has been made because of inadequacy of the data. All measures of total production appear to have a "downward bias."[24] The most adequate measurements depend upon figures obtained by the Census of Manufactures. The writer has shown elsewhere that the coverage of the Census is subject to an undetermined variation.[25] This is indicated by a large variation in the number of companies canvassed by the Census from 1927 to 1933, entirely unexplained by published statements. A further difficulty is that the Census adds new industries very tardily and, hence, coverage is incomplete for a part of total industry which grows very rapidly. With such limited knowledge, it is not useful to attempt measurement of the long-time trend of total industry at the present time. Probably the best available conclusion regarding the past growth of total industry is that

[23] Outstanding advocates of this position are W. W. Hay, L. E. Peabody, R. B. Prescott, and Simon Kuznets. See particularly W. W. Hay, "Study of the Nature of Demand Would Obviate Many Mistakes of Management," *The Annalist*, May 22, 1931; L. E. Peabody, "Growth Curves and Railway Traffic," *Journal of the American Statistical Association* 19: 476–83, December, 1924; R. B. Prescott, "Law of Growth in Forecasting Demand," *Journal of the American Statistical Association* 18: 471–79, December, 1922; Simon Kuznets, *Secular Movements in Production and Prices*, 1930.

A useful analysis of the Gompertz curve, comparing it to the logistic, will be found in C. P. Windsor, "The Gompertz Curve as a Growth Curve," *Proceedings of the National Academy of Sciences* 18: 1–8, January 15, 1932; see also G. R. Davies and W. F. Crowder, *Methods of Statistical Analysis in the Social Sciences*, 1933, pp. 320–24.

The equation of the logistic is $Y=k/(1+e^{a+bx})$, and of the Gompertz is $Y=ab^{c^x}$. Both curves may be generalized; for the Gompertz see the Windsor reference, and for the logistic see Raymond Pearl, *Studies in Human Biology*, 1924.

Particularly significant general analyses of these methods are A. B. Wolfe, "Is There a Biological Law of Human Population Growth," *Quarterly Journal of Economics* 41: 557–94, August, 1927; E. B. Wilson and Ruth R. Puffer, "Least Squares and Laws of Population Growth," *American Academy of Arts and Sciences Proceedings* 68: 286–382. Further references will be found in the Windsor reference given above and in A. B. Wolfe, "The Population Problem Since the World War: A Survey of Literature and Research," *Journal of Political Economy*, volumes 36 and 37.

[24] The reasons for this downward bias are well stated in A. F. Burns, *op. cit.*, pp. 17–27, 258–259.

[25] In an unpublished study made under the auspices of the Central Statistical Board. For given products, variation reaches truly remarkable proportions. Pyroxylin is a notable case.

which Burns has established with great care: "if there has been any decline in the rate of growth in the total physical production of this country, its extent has probably been slight, and it is even mildly probable that the rate of growth may have been increasing somewhat."[26] We may use such a conclusion as a working hypothesis.

It is well to point out the conformity of Burns' conclusion with the fact that practically every industry has been growing at a decreasing rate. Such conformity is clearly dependent upon a good distribution of industries as to development. If, for instance, total industry were made up of two industries, one mature and growing slowly, the other young and growing rapidly, total industry might grow as shown in the following hypothetical situation:

Industry	*Hypothetical production in the stated year*		
	1	*2*	*3*
A	100	101	102
B	1	2	3.5
Total	101	103	105.5

It is obvious that each of the series in this hypothetical illustration is increasing at a decelerating rate, but that the total increases 1.98 per cent from year 1 to year 2, and 2.43 per cent from year 2 to year 3. Such a mathematical relationship is logically possible since the amount of growth of the small industry is so much greater from year 2 to year 3 than from year 1 to year 2 that it produces an increased percentage on the larger base.

It is possible in actual practice, therefore, that the amount of growth of small industries may be so great that, when added to the level of the more important industries, an accelerating rate of growth results even though the rate of growth in each industry is decelerating. To accomplish this result, the small industries must show an acceleration greater than an arithmetic series, but need not accelerate as rapidly as a geometric series. If we assume that total industry is well divided between large and small industries, and that the small industries are growing at relatively large amounts, it is easily conceivable that total industry may be growing at a constant rate. In addition, the tendency for declining industries to level off will contribute to the maintenance or acceleration in the rate of growth.

We have no accepted equation descriptive of the long-time trend of total industry. Such an equation cannot be established empirically, at the present time, because of inadequacies of the data. The most adequate analysis yet made leads us to believe that, in the recent past, total industry has been growing by approximately a constant

[26] *Op. cit.*, p. 279.

rate, in spite of the fact that almost every one of the component industries has been growing by a decelerating rate.

(F) FORECASTING THE LONG-TIME TREND

Because the analysis of economic growth has been restricted principally to an attempted statement of the functional law of growth, forecasting of the long-time trend typically has been thought to involve the following procedure: (1) establishment of the law of growth; (2) measurement of the trend growth by fitting the equation of the law of growth to the data; (3) extrapolation of the trend measurement found under (2) to be used as the forecast.[27] Since economists have found no generally accepted law descriptive of the growth of total industry, it is not possible to forecast it by such a procedure. This no doubt accounts for the skepticism noted at the beginning of this paper.

The writer believes it feasible to attempt forecasts of the long-time trend without the use of an established law of growth. By fitting a wide range of equations to varying periods and comparing the results, it should be possible to make a reasonably definite measurement of the long-time trend of the past.[28] By analysis of the factors determining the long-time trend, it should be possible to establish what modifications should be applied to the extrapolation of the accepted trend in order to make it conform with the most reasonable expectations.[29]

Since, as noted in the preceding section, it is not possible to obtain a reasonably adequate measurement of total industry at the present time, we cannot carry out the first part of this procedure. To carry out the second part, it is necessary to state and analyze the logical factors determining the long-time trend of total industry. Stating these factors is a pioneer problem, since there has been but little analysis made of the subject.[30] The writer believes the significant factors to be as follows:

[27] C. L. Knight seems to state the typical concept. "When the subject-matter of the series is governed in its growth or decline by ascertained laws, these afford a basis for the projection of a defensible line of trend." *Secular and Cyclical Movements in the Production and Price of Copper*, 1935.

[28] In an unpublished manuscript, the writer has made a detailed statistical study of this sort for the steel industry.

[29] It is true that these factors are involved in the measurement of the past trend, but this total measurement blurs, by combination, significant influences present in the separate factors. Recognition of the necessity of considering each factor separately is found in the Thompson and Whelpton forecasts. See *Population Trends in the United States*, 1933.

[30] W. C. Mitchell would classify the causes of secular trends as "(1) causes related to changes in the number of population; (2) causes related to the economic efficiency of the population—its age, constitution, health, education, technical knowledge and equipment, methods of cooperation, methods of settling

(1) Population level and growth
(2) Ratio of the gainfully employed to the total population and any shift in the quality of the employed people
(3) Average length of the working week for labor and capital
(4) Nature of the demand
(5) Technological ability and changes in this ability
(6) Extent new capital is put to the most economic uses
(7) Savings
(8) Condition of natural resources
(9) Efficiency of selling and distribution
(10) Violence of the cyclical movement
(11) Extent to which profitable exchanges are permitted with foreign countries
(12) Any permanent or semi-permanent lack of balance

Within the limitations of an article of this sort, it will not be possible to make a very complete statement of the most probable influence of each of these factors in the past or of the most probable influence of each in the future. Further, the relation of these factors to the long-time trend has never been faced squarely enough for a very complete knowledge to be available. Brief summary statements follow.

There is a marked deceleration in the rate of population growth at the present time, and there is a clear indication that this slowing down will continue into the future.[31] This slower population growth is dependent upon a marked decrease in the birth rate, which seems to be due principally to a more effective exercise of volition through the wide use of contraceptive devices. It is clear that this slowing down of the population growth will shift the type of goods demanded because of a radical change in the age distribution, but in spite of the relatively common belief that this slower growth in population will result in a slower growth for total industry, the writer holds that the probable influence over the next fifteen to twenty years will be small. The present population level appears to be clearly above "optimum."[32] This

conflicts of interest, and many other matters; (3) causes related to the quantity and quality of natural resources exploited by the population." *Business Cycles*, 1927, p. 231.

In American Industry and Commerce, 1930, pp. 99–103, E. D. Durand lists "the most basic of the causes that have made American economic life what it is." He lays predominant emphasis upon natural resources and character. Durand attempts chiefly to show the reason for the high *level* of the trend compared to that of other countries, while the chief objective of the present article is to indicate the chief factors responsible for the growth of the trend.

[31] See W. S. Thompson and P. K. Whelpton, *Population Trends in the United States*, 1933.

[32] "Optimum" population levels are defined as those at which a maximum

is indicated by the belief that there were a number of unemployed people willing to work in the last phase of prosperity, i.e., over and above purely technological unemployment, and by the belief that unemployment will be a characteristic of the next prosperity.[33] It might seem that, with a slower population growth, we would be coming closer to "optimum" levels. But this is not necessarily true, since, with slight rearrangements of our machinery, industrial capacity was greatly increased in the past decade.[34] Further evidence of above optimum population levels is furnished by the fact that people are restricting the size of their families in order to maintain a higher standard of living. On the average, we will be unsuccessful if this effort does not increase the per capita output (considered as a "crude rate" with the total population, regardless of age, as the denominator) at least for a period of some years into the future. With a rapid decline in the birth rate, this should be true, of course, whatever the purpose of the population restriction, since the very young necessarily are an economic liability. For a few years, a decrease in their number merely decreases the number of dependents without affecting the number of workers. If we have a population which is above "optimum" in size, it may be that population restriction will have no adverse effect whatever on *total* production.

The larger the ratio of the gainfully employed to the total population, the greater is the supply made possible. Over the last thirty years this ratio has remained about constant. This has taken place by an offset between a decrease in children gainfully employed and an increase in women gainfully employed. Due to the technicalities of the definition,[35] a better indication is given by the fact that the proportion of the total males employed has remained about constant. Probably the tendency for women to engage in economic activities will continue, probably the employment of children will decrease still further, and possibly the proportion of men employed will decrease due to

output per capita is made possible. See A. M. Carr-Saunders, *The Population Problem: A Study in Human Evolution*, 1922.

[33] This increase in unemployment may be due, however, partly to a natural increase in immobility which accompanies the onset of stationary or declining population. See Edwin Cannan, "The Changed Outlook in Regard to Population, 1831–1931," *The Economic Journal* 41: 519–32, December, 1931.

[34] See Harry Jerome, *Mechanization in Industry*, National Bureau of Economic Research, 1934.

[35] Housewives are excluded from the count of the gainfully employed. To some extent, therefore, the employment of women erroneously increases the count of the employed. A housewife may go into economic pursuits and hire a maid to perform the work she formerly did. If the housewife is more valuable in economic pursuits, however, this may increase somewhat the total productive ability of the labor force.

increasing economic independence.[36] There is no clear indication that the ratio will change much in the near future. A shift in the quality of the employed people may take place because of the average age of the employed or because of the type of preparation the employed receive. We know but little about the effect of age or of various types of preparation.

There has been a shortening of the working week for labor and probably for capital.[37] Our knowledge of the effect of fatigue is meagre, so the net influence of the shortening of the working week is difficult to state. Since the decline has been rather marked in the past, perhaps the deleterious effect in the future will be no greater than it has been in the past.

Due to scarcity of economic goods, we assume that there is, on the average, a demand for whatever total quantity of goods the economic organization is capable of producing. This does not imply an effective demand for the total goods which could be produced by our mechanical capacity for, as noted above, the economic organization is not capable of producing at this peak. Economic demand may be for goods or for services. Service industries have grown rapidly during the past decade, and there seems to be little reason for believing that their trend relative to that of goods industries will grow more rapidly in the future. Within limits, society may choose to take more goods or goods of higher quality.[38] A similar choice is that between more goods or goods of better appearance and those used more conveniently. Superficially, at least, it appears that society made large choices in favor of higher quality, better appearance, and greater convenience, in the last decade. It would appear improbable that such choices would be relatively any greater in the near future.

(G) INFLUENCE OF CAPITAL AND TECHNOLOGICAL CHANGES ON THE TREND OF TOTAL PRODUCTION

Important changes have been taking place in our technological ability to produce goods, as shown by the rapid increases in output-

[36] There has been a slight decline in the ratio of males gainfully employed above the age of 16.

[37] The working week for capital does not necessarily decrease with a decrease in the working week for labor, as for instance, is the case in the steel industry. In many industries, however, there probably has been a decrease in the working week for capital. It is obvious that such a decrease results in (1) a decrease in effective capacity, (2) an increase in depreciation due to natural elements, and (3) an increase in the period capital must be invested to produce a given return.

[38] To some extent, goods of higher quality are the result of mechanical improvements and can be produced as cheaply as goods of inferior quality. This is true, for instance, in the case of many young consumers' goods industries, such as the radio a few years ago.

per-man-hour. Such rapid increases certainly have greatly increased our mechanical capacity to produce goods, and according to the evidence adduced earlier that the long-time trend level tends to bear a more or less constant proportion to this capacity, it would seem that the increase in technological ability has had an important effect on increasing the level of the long-time trend.[39] If technological improvements occur too rapidly, however, their full effect in raising the long-time trend may not be felt immediately. The lag between the time technological changes become scientifically practicable and the time they are commercially applied may become exceptionally long. If technological improvements occur with such rapidity, a "margin of safety" is set up for the future, so that insofar as technological improvements are concerned, the long-time trend may increase as rapidly in the future as in the past, even though there may be some decline in the rapidity with which our technological ability is improved.[40]

The extent to which new capital is put to the most economic uses depends upon the facility with which economically sound investments are made.[41] The more capably long-time trend forecasts are made, the better will the new capital be placed. It is common knowledge that much capital has been wastefully and fraudulently placed in the past. With present attempts to curb fraud in the selling of securities, as exemplified by activities of the Securities and Exchange Commission, it is reasonable to expect that there may be some improvement in the extent to which capital is put to the most economic uses in the future.

Moulton has shown that capital investment depends principally upon demand for consumption goods.[42] He seems to hold that the creation of capital goods does not involve a diversion of capital and labor from the production of consumption goods. If this were true, the amount of savings would have no determining influence on the long-time trend of production. Moulton notes that the production of capital and consumption goods are positively correlated. The crucial portion of his demonstration, however, depends upon the fact that

[39] In this assumption that the long-time trend tends to bear a more or less constant proportion to the capacity to produce there are involved also (1) the violence of the cyclical movement and (2) factors of unbalance. These determinants are considered below. An increase in technological ability raises the ability to produce and, therefore, potentially raises the trend level. It is proper to consider separately whether this is partly or wholly offset by increased violence of the cyclical movement or by an increase in unbalance.

[40] That such excessive technological improvement may lead to unbalance is another matter.

[41] There are cases where the technological question may be as important, as e.g., in the decision to place mining machinery at a given point.

[42] *Formation of Capital*, 1935.

savings increased faster than consumption in the decade of the nineteen-twenties, while actual local investments did not increase proportionately with savings. He accounts for the difference between savings and investments by foreign loans and by the bidding up of security prices. Now the investments resulting from foreign loans should be added to the investment side *or* the foreign loans should be subtracted from the savings side before a proper comparison can be made. The fact that most of these foreign loans may be a dead loss has nothing to do with the problem. The fact that a large part of savings were "dissipated in bidding up the price of outstanding securities" is not impressive. For, a large part of the savings *originated* from the bidding up of security prices.[43] If there had been less bidding up of security prices, there would have been less savings originating from the bidding up of security prices. It would be surprising if such a dissipation of savings were not the characteristic of financial boom periods in the first half of the nineteenth century, during which time Moulton feels "the volume of funds available for the purpose of capitalistic enterprise was persistently inadequate." Therefore, the fact that 15 billions of investment money were available in 1929 while only 5 billions "passed into the hands of business enterprisers [in the United States] for use in buying materials and hiring labor for the construction of new plant and equipment" is not conclusive evidence that capital construction is unrelated to savings in prosperity.[44]

There comes naturally to mind the analysis of cyclical movements by Keynes and Hayek.[45] These writers are in agreement in holding that the business cycle is to be explained in terms of a disparity between savings and investments. While interesting from this point of view, their analyses do not bear on the problem of the extent to which

[43] The study on *America's Capacity to Consume* by Maurice Leven, H. G. Moulton and Clark Warburton, 1934, indicates that capital gains of this sort were over 6 billions in 1929.

[44] Even the facts cited cannot be accepted as at all representative, however. In an analysis made for the National Bureau of Economic Research, Kuznets arrives at investment figures very different from those found by Moulton. See Simon Kuznets, *Gross Capital Formation 1919–1933*, Nat. Bur. Econ. Res. Bulletin 52, November, 1934.

[45] The best expositions of their positions will be found in J. M. Keynes, *A Treatise on Money*, 1930, and F. A. Hayek, *Monetary Theory and the Trade Cycle*, 1932. Excellent criticisms of their positions will be found in Alvin H. Hansen and Herbert Tout, "Annual Survey of Business Cycle Theory: Investment and Saving in Business Cycle Theory," ECONOMETRICA 1: 119–47, April, 1933; Paul Douglas, *Controlling Depressions*, 1935, pp. 41–52; Jack Stafford, "A Note on the Equilibrium Rate of Interest," *The Economic Journal* 45: 259–68, June, 1935.

savings determine the growth of the long-time trend of total production.[46]

Since the potential production of consumption goods is restricted by the current production of capital goods,[47] it is clear that the amount of savings is a principal determinant in the production of durable goods.[48] The accumulation of capital goods depends upon the amount of savings. Since savings are required to make use of technological advances, and since increased quantities of capital ordinarily make for more efficient production, savings occupy an important place in the factors making for an increase in the long-time trend. Looked at in a more general fashion, the factors of production are population, capital accumulations and savings, and natural resources. The importance of each of these factors is determined in general by the law of proportionality. The effectiveness of the combination is also conditioned by our technological ability. It would appear that savings represent the most critical factor in the combination since population apparently is above the "optimum" level.[49]

(H) EFFECTS OF OTHER FACTORS ON EQUILIBRIUM POSITIONS

The effect of the other factors determining the long-time trend of total industry can be stated only in a very tentative fashion. Economists have given but little attention to them as trend determinants.

The condition of natural resources probably is slowly deteriorating. However, there is no good reason to believe that the rate of deterioration will be any greater in the near future than in the recent past. Probably the condition of natural resources can be accepted as a

[46] The fact that a more violent cycle produces a lower long-time trend is another matter. It is interesting to note that Moulton carries the analysis over into the long-run tendencies from a mandate by the Falk Foundation. See *Report of the Maurice and Laura Falk Foundation of Pittsburgh, Pa., for 1933 and 1934*, part two, pp. 1–2.

[47] See section (D) above.

[48] This statement does not deny the fact that capital goods, at times, are created by credit at the expense of the general purchasing power. Ordinarily such creation cannot directly affect the long-time trend, however, whatever its effect may be on the cyclical movement.

[49] Paul Douglas attributes the increase of production to the growth in labor and to the growth in capital, and conceives the effect of each as inversely proportional to its rate of growth. Such a procedure makes labor a much more important determinant than savings. See Paul Douglas, *Theory of Wages*, 1934, pp. 131 ff.

Clark points to the need for a balance between savings and the uses to which savings may be put. See J. M. Clark, *Strategic Factors in Business Cycles*, pp. 148–49.

passive factor in the determination of the long-time trend.[50] Where the rate of exploitation is rapidly increasing, such a conclusion cannot be accepted, but there appears no longer to be a marked increase in the rate of exploitation of natural resources in the United States. A critical point will occur for given industries if a crucial raw material gives out, but unless such a situation can be anticipated, there is little reason why it should give us much concern when related to total industry for short periods into the future.

The efficiency of selling and distribution represents our technological ability to distribute goods. Probably it is correct to measure this technological ability by output-per-man-hour figures. Such a conception disregards entirely any advertising service performed in educating the clientele as to the uses and relative advantages of various goods, but the net value of such service is very uncertain. It appears that such technological ability can be counted on to be somewhat greater in the near future than in the recent past. With the advent of chain stores and similar distributive and selling methods, many less productive methods have hung on, and because of the increased competition, they have become less productive. In transportation, a somewhat similar situation exists with the development of the truck for short hauls, while unprofitable branch lines of the railroads have been kept in use. As economic society discards the outmoded schemes, the efficiency of the more efficient types will be more nearly representative of the total.[51]

The greater the violence of the cyclical movement, the lower the proportion the long-time trend is of the physical capacity to produce and the lower, therefore, is the level of the long-time trend. At the present time, we have no clear basis for anticipating the future violence of the cyclical movement. It may be greater in the future due to the increasing proportion of "dispensable" goods in our standard of living. The purchase of durable consumers' goods, which are "dispensable"

[50] S. H. Slichter seems to maintain an opposite position. "Our natural resources are pretty well developed and the flow of immigration has ceased. Unless industry receives an extraordinarily strong stimulus from technological discoveries, it will grow more slowly in the future." See *Towards Stability*, 1934, p. 77. It scarcely seems that the fact that our natural resources are pretty well developed should mean a slackening of the trend growth in the near future, since in the recent past the development and exploitation of natural resources was about what can be expected in the near future.

Durand holds that natural resources are a chief factor in the level of production. See footnote 30 above. Such a position is consistent with the one presented, because our analysis is focused on factors determining growth rather than level.

[51] The *direct* advantage of some improved types of distribution seems to be reduced somewhat by an increase in advertising costs entailed.

in terms of the cyclical movement, is becoming proportionately larger. Other factors, such as the stage of industrialization,[52] the rapidity of long-time changes, the diverging movement in long-time changes,[53] and the amount of unbalance existing in society, may be tending to increase the violence of the cyclical movement. It is worth noting that the only way one severe depression can greatly alter the long-time trend level is by setting up disruptive structural changes which produce unbalance.

The greater the freedom permitted in making foreign exchanges, the higher is the level of the long-time trend of total industry. Artificial foreign trade restrictions, as typified by bilateral pacts and high tariffs, amount to an important threat. The full effect of such devices may not have been felt; and since there is no clear basis for expecting any decrease in these restrictions,[54] foreign trade may be expected to have a future trend somewhat lower than indicated by extrapolation of the past growth. This factor can be over-emphasized, however. A large proportion of the products we need can be produced in the United States at a not very unreasonable cost, and we have within our borders a wide and diversified market. Further, on the whole the restrictions do not prevent us from obtaining from without goods which we would have to produce at a ridiculous cost. Principally, they increase somewhat the cost of many products and restrict the foreign market.

"Any permanent or semi-permanent lack of balance" technically is an incorrect appellation. These factors do not prevent balance; they prevent return to the former balance. However, all structural changes operate in this way. The difference lies in the fact that, in the case of factors creating "lack of balance," the new hypothetical balance is imposed by restrictions of illogical nature. Price rigidity is a chief factor of this sort. Whether existing tendencies will produce an increasing rigidity of prices is a question which has not been given adequate analysis. It appears that an institutional shift resulting from the present depression may have a deleterious effect on the long-time trend. A large body of people have been unemployed long enough to make it appear possible that their habits of thought may have shifted so that, when work is again offered, they may not freely offer their services.

[52] See F. C. Mills, "An Hypothesis Concerning the Duration of Business Cycles," *Journal of the American Statistical Association* 21: 447–57.

[53] See F. C. Mills, *Economic Tendencies*, 1932, pp. 10–11, 62–63.

[54] The chief hope for United States' foreign trade appears to lie in reciprocal trade agreements, but these do not appear to be progressing very rapidly. A good analysis of the foreign trade situation will be found in *Report of the Commission of Inquiry into National Policy in International Economic Relations*, 1934.

(I) SUMMARY

The writer would classify as follows the past influence of these factors determining the long-time trend of total industry:

Factors which have exerted large influence in making for an upward growth
(5) Changes in technological ability
(7) Savings

Factors which have exerted a mild influence in making for an upward growth
(1) Population growth
(9) Efficiency of selling and distribution

Factors which have exerted no appreciable influence in making for an upward growth
(2) Ratio of the gainfully employed to the total population
(6) Extent new capital is put to the most economic uses
(8) Condition of natural resources
(10) Violence of the cyclical movement
(11) Extent to which profitable exchanges are permitted with foreign countries
(12) Any permanent or semi-permanent lack of balance

Factors which have exerted a negative influence on the upward growth
(3) Average length of the working week for capital and labor
(4) Nature of the demand

It would appear that factors (6), extent to which new capital is placed in the most economic uses, and (9), efficiency of selling and distribution, will exert a more beneficial influence on the future long-time trend than they have on the past. Factors (10), violence of the cyclical movement, (11), extent to which profitable exchanges are permitted with foreign countries, and (12), any permanent or semi-permanent lack of balance, are expected to exert a more deleterious effect on the trend of the future than on that of the past. The writer would conclude, tentatively, that the future long-time trend of total industry can be expected to be slightly under the approximately constant rate of growth of the past, set up as a working hypothesis. It is to be hoped that qualified students may find it possible to make sustained analyses of the determining influence of these various factors and modify this result to a narrower range of probability.

The scheme presented makes possible an analysis of dynamic economic change. By following this technique, we can state the most probable position of the shifting equilibrium. One need for such a statement, emphasized in this paper, is the forecast of the long-time trend. A rea-

sonably adequate forecast of the long-time trend should make possible the building of capital equipment on the basis of the trend indication, instead of according to the misleading indications of the current demand for the usufruct. Since the major fluctuation over the business cycle is in the production of durable goods, this should, if carried out on a wide scale, at least to some extent reduce the cyclical violence.

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MEETINGS OF THE ECONOMETRIC SOCIETY IN NEW YORK CITY, DECEMBER, 1935 AND ST. LOUIS, JANUARY, 1936

THE AMERICAN MEETINGS of the Econometric Society were held in New York on December 30–31, 1935, with the Social Science Societies, and in St. Louis on January 2–3, 1936, jointly with Section K of the American Association for the Advancement of Science.

The opening session of the New York meeting was held in the Commodore Hotel, on Monday morning, December 30, the general topic being "Economic Equilibrium and Stability." Professor Wassily Leontief of Harvard University in the first paper, "Particular Equilibrium and Discontinuous Supply Adjustments," presented a theoretical analysis of the so-called "cobweb" problem, where the supply adjustment is discontinuous and lagging behind the demand adjustment, which is continuous. He pointed out that starting with an initial produced quantity q_0, and a corresponding demand price p_0, the subsequent supply adjustment brings the production up to q_1 and the price down to p_1; the ensuing contraction in output reduces the supply to q_2. The further process of staggering supply adjustments will proceed along similar lines, showing either (a) a "centripetal" tendency with diminishing amplitudes, or (b) a centrifugal development with increasing amplitudes, or (c) equilibrium, an exact repetition of the first cycle. The three possible cases are defined by the value of the difference, $D(q_0)$, between the initial and final quantity of the first cycle. For all values of q_0 less than Q, the "Marshallian" equilibrium quantity, $D(q_0)>0$, means a centripetal tendency, $D(q_0)<0$, a centrifugal tendency, and $D(q_0)=0$, equilibrium. The equilibria are stable, unstable, or neutral, according to the sign of the first derivative of $D(q_0)$, and additional characteristics are obtained by consideration of the second derivative. Further analysis discloses a definite correspondence between the relative position of the different equilibrium points and their stability characteristics.

The second paper of the session, "The Theory of Choice and the Determinateness of Economic Equilibrium," was presented by Dr. Nicolas Georgescu of the Rockefeller Foundation, who made a distinction between the conditions of stability and those of equilibrium, pointing out that Allen's conditions did not represent stability. He showed how equilibrium is reached in the case of three commodities and gave results regarding the saturation region. Following a re-examination of Pareto's mental experiment leading to the well-known equation $\sum_x \phi_x dx=0$, a range of indeterminateness of choice was introduced. Dr. Georgescu then discussed conditions of stability in the new

problem and various consequences regarding the demand curves, static equilibrium and the postulates proposed for the measurability of utility. A third paper was given by Mr. Dana Raymond, whose subject was "Stability under Cooperative Systems."

On Monday afternoon a joint session was held with the American Statistical Association at the Biltmore Hotel under the general subject "Statistical Technique." Professor Harold Hotelling of Columbia University, who opened the meeting with a paper "Generalized Multiple Correlation for Pairs of Sets of Economic Variates," recalled that he had previously presented a paper on some relations between two sets of variates which are invariant under linear transformations of the variates of either set. He explained that the invariants under these internal transformations are the roots of a certain determinantal equation in which the correlations among the variates appear, together with functions of these roots. This determinantal equation occurs in the problem of finding linear functions of the two sets possessing maximum correlation with each other; this maximum correlation is obtained from the greatest root of the equation. Certain applications of these invariants in economics were suggested—one to index number theory, another to correlation of time series with variable lag, and another to the statistical evaluation of supply and demand functions. For instance, instead of studying the relation of crop to price of a single agricultural commodity, one may, with profit, treat a group of related commodities together, as has been done statistically by Henry Schultz in the *Journal of Political Economy* for August, 1933, and theoretically by Harold Hotelling in ECONOMETRICA for January, 1935. In these circumstances the invariants under internal transformations in relation to their probability distributions serve to test various hypotheses that may be made, for example, as to the effects of a taxation and crop-restriction program.

Professor Walter Baude of the University of Cincinnati read a paper "The Significance of Points of Inflexion in the Determination of Cyclical Variation," following which V. S. von Szeliski of Washington, D. C., presented "Some Problems in Time Series Analysis." After these papers, Professor S. S. Wilks of Princeton University led a discussion on the theoretical significance of the papers while Mr. Robert Burgess of the Western Electric Co. spoke of their practical importance.

A Monday evening dinner session was held at the Roosevelt Hotel with Mr. Stephen DuBrul of the General Motors Corporation presiding. Mr. John P. Scoville of the Chrysler Corporation discussed the "Behavior of the Automobile Industry in Depressions." This paper has since been printed in pamphlet form and may be obtained from the Chrysler Corporation.

Tuesday morning, a session presided over by Professor Irving Fisher was held at the Commodore Hotel, the general topic being "Monetary and Exchange Problems." Erling Petersen of the University of Oslo, Oslo, Norway, presented the first paper on "Some Aspects of the Equation of Exchange, Especially in Relation to Replacement Rates." He said that Lubbock, in 1840, treated the exchange system as dynamical, which requires more complex expressions than the simplified forms like the quantity theory. Since the quantities in the $PT=MV$ equations are not homogeneous, it suggests further decomposition, and the following double equation was considered: $Pg(1+\mu)(1+\phi)i = Mx(1-\psi)(1+\phi) = s(vw+p)(1+\phi)(1+\phi)$, the factor ϕ representing intermediary goods per unit of consumer-bought product and varying with replacement rates of depleted stocks and worn out machinery. An analysis of the equation shows that saving-investment equality is not a sufficient condition for equilibrium, and tentative calculations show important cyclical fluctuations in ϕ.

A second paper was read by Professor Richard Bissell, Jr. of Yale University, entitled "Comments on The Theory of Capital." Professor Bissell in discussing "circular relationship" of the Austrian school and its effect on the average length of the period of production, showed that this average period of a producers' good can be defined as the quotient of two infinite series to give a value slightly greater than if the "circular relationship" were ignored. He then commented on the concept of the anticipated period of production. Making use of an expression for the average elapsed time between investment and output, which he developed, Professor Bissell deduced two conditions of equilibrium which must be satisfied if present profits of a firm are to be maximized. These two conditions, together with the production function, would be sufficient to determine the total output, the average period of investment, and the amount of labor that would be employed.

The third paper was given by Oskar Lange of the University of Krakow, Poland, whose subject was "Interest and Capital Formation." Mr. Lange developed a short period theory of interest from a general production equation and showed that the rate of interest is equal to the marginal net productivity of indirect labor.

At 11:30 Tuesday morning the session on "Statistical Methodology" was presided over by Professor Harold Hotelling of Columbia University. Mr. N. D. Roodkowsky spoke on "Statistical Induction." He said, in part, "A survey of the writings of the nineteenth century logicians shows that the basic assumption of induction was the notion of cause derived from an uncertain mode of inference called induction per enumeration of instances. Although holding absolute certainty as its aim, the traditional inductive theory could not advance logical

justification of its universal propositions, which could be invalidated by one instance to the contrary."

"On the other hand, the statistical generalization, expressed in the form of infinite hypothetical population, is not discarded if the second sample is significantly different from the first sample which was considered as belonging to the above population. This difference implies only that the co-existential matrices, which gave rise to the samples, are essentially distinct with respect to the attribute in question. In fact, formal statistical propositions cannot be absolutely verified or refuted by experiment. They are similar in nature to Poincaré's hypotheses, which can neither be proved nor disproved by experience."

After discussing Fisher's "likelihood" and "sufficient" statistics, Professor Roodkowsky continued, "Statistical generalizations describe fundamental relations and co-existences of the physical world, which become less uniform when we study the more minute elements in nature. Therefore, the statistical generalizations must be considered as primal inductions, whereas the observed uniformities, or so-called laws, are derived from the basic statistical populations. This does not mean that all the practical rules of the traditional theory of induction are vitiated. We still deal with events consisting of large masses of very small elements whose individual behavior is contingent. Hence we observe regularities which can be analyzed by modified canons of prestatistical induction."

"Price and Demand" was the general topic considered Tuesday afternoon in a session over which Mr. Louis Bean presided. Here Mr. R. H. Whitman of R. H. Macy and Co., New York, presented a paper, "Demand Study Technique Applied to Department Store Data," saying, in part: "Department store customers react to short-run fluctuations in prices, but current sales are a function of past as well as present prices. To obtain a demand function the usual procedure is to collect data on the price and sale of an item for a period of two to three months and fit them to some form $y=F(p, t)$, or as an alternative $y_1/y_0 = F(p_1/p_0)$. If the second form is used, the demand relation may be generalized for a group of similar commodities. Experience shows that such generalizations have prediction value as customers' reactions change infrequently. They, as distinct from Marshallian functions $y=F(p)$, or more generally $y=F(p, t)$, are valuable in determining department store sales policy. Further progress depends on the accumulation of sufficient information on customer behavior under the more complex situations, and development of a more complete technical analysis."

Mr. von Szeliski gave the next paper, entitled "Economic Behavior of the Cotton Textile Industry." He said that in spite of the steady

increase in national income during the 20's, the gross income of the cotton textile industry steadily decreased until 1933. A correlation analysis of production with the size of profit margin and annual price changes showed that they were approximately of equal importance. A study of consumers' reactions demonstrated that the national income was more important than price changes. Production in previous years and price changes account for year to year fluctuations in production while national income and relative prices account for the major cycles.

On the same afternoon, under the general topic "Unemployment," Professor Emil Lederer considered "The Problem of Technological Unemployment." He disputed the assumption that reduction in the wage bill, due to a technological improvement needing additional investment, will exactly equal the increase in profits or consumers' purchasing power, as the total income will shrink if only savings are invested. If profits increase, an increase in investment may lead to reemployment of part of the dismissed workers, but the reemployment of all the workers will take considerable time and be feasible only at a lower wage rate. If the amount of capital available for investment can be increased, the additional purchasing power may increase prices, leading to further demands for capital; and thus the technological improvements may become one of the factors causing over-investment and the resulting depression. Capital-saving devices may also cause unemployment of both workers and capital, creating the difficult situation of combining idle capital and labor into new units of production.

The concluding paper of the New York meeting was presented by Mr. Nicholas Kaldor of the London School of Economics on "Wage Subsidies as a Remedy for Unemployment." He said an examination of the relative methods of reducing unemployment leads to the result that subsidy on wages yields larger additional employment per unit of cost than either public works or a subsidy on production, and is likely to stimulate employment to a greater extent than an equivalent percentage reduction in money wages. The easiest form of administration is to remove those forms of taxation which can be considered a negative wage subsidy. The net effect will be to increase the total income of labor and the community. As a long run policy the scheme is only feasible if money wages can be stabilized; if the subsidy is so large that the wage level rises, it might reduce the income of capital and defeat its own end.

* * * * *

The first of the meetings of the Econometric Society in St. Louis was held the morning of January 2, 1936, the topic being "Mathematical Economics and Statistics." In the principal paper, "Mathematical

Theory of Index Numbers," Dr. Thomas H. Rawles of Colorado College, after reviewing briefly the index number tests introduced by Fisher, adopted Divisia's definition as his point of departure. Several methods, including that of René Roy, were given for the approximate integration of the equations involved, and it was shown how Fisher's "Ideal" number can be derived from Divisia's hypothesis. Since for arbitrary data the circular test is not satisfied by numbers obtained from this definition, additional sufficient restrictions on the data were developed to enable this test to be satisfied. Suggesting that economic systems in which the data conform to these restrictions be called, "Conservative Economic Systems," Dean Rawles interpreted the amount by which the circular test fails in terms of the relation between the variation of prices and quantities. In order to expose the nature of the difficulties which cause the failure of the circular test, he suggested the advisability of introducing as an approximation to the utility function a quantity function which is homogeneous of degree one in the quantities of the commodities. He concluded by discussing some recent attempts to apply the "method of limits," followed by some general remarks on the practice and theory of index numbers.

Francis Regan of St. Louis University spoke on "The Admissibility of Time Series." He said in a previous paper (*Transactions*, Vol. 36, 1934, pp. 511–29) he had constructed a time series in such a manner that the number $x(\alpha, \tau, t, \Lambda)$ is an element of the set $A[f(\alpha, \tau, t)]$ for every α, τ, t and Λ, where $\tau = m\cdot 2^{-\sigma+1}$, $t = r\cdot 2^{-\sigma+1}$, $\Lambda = \rho 2^{-\sigma+1}$ and $r+m \leqq \rho$; and where α, ρ, m, r and σ are positive integers. He went on to show that for the same series the number $x(\alpha, \tau, t, \Lambda)$ is a member of the set $A[f(\alpha, \tau, t)]$ for every α, τ, t and Λ, where $\Lambda = \rho 2^{-\sigma+1}$ and $t+\tau \leqq \Lambda$, with α, ρ and σ as positive integers.

Solomon Kullback of Washington, D. C. spoke on "Certain Distributions Derived from the Multinomial Distribution," followed by Nelson Norris of the University of Michigan who discussed "Convexity Properties of Generalized Mean Value Functions." W. D. Baten of the University of Michigan then read a paper on "The Frequency Distributions for the Mean of n Independent Chance Variables when Each is Subject to the Law, $y_0 x^{p-1}(1-x)^{q-1}$."

In the afternoon, Professor Harold T. Davis of Indiana University presented a paper "Statistical Validity of the Forty-Month Cycle in Stock Prices," in which he considered the problem of establishing the reality of the harmonic components by means of probability. The permanence of the cycle was first exhibited by means of a Schuster harmonic analysis of the Dow Jones averages in the two periods: 1897–1913 and 1914–1924; by a harmonic analysis of the Cowles index of stock prices from 1880 to 1897; and by an analysis of rail stock prices

from 1831 to 1855. In order to show the statistical validity of the harmonic component the method of R. A. Fisher (*Proc. Royal Soc. of London*, Vol. 125 (A), 1929, pp. 54–59) was employed. This test showed a probability in excess of .95 that the amplitude of the periodogram in the neighborhood of 40 months was significant. The pattern was effaced by an abnormal change in trend lines during the period of the American Civil War and during the bull market of the 1920's. The second part of the paper considered the dynamics of the series. It was shown that if the 40-month cycle could be accounted for by an impressed force of the form $E \sin pt$, $p=2\pi/40$, then the differential equation

$$Ay'' + By' + Cy = E \sin pt$$

might not only account for the existence of the cycle, but also show why the pattern was effaced during the two periods of stock inflation. A slight variation in p might make $C-Ap^2$ zero and lead to the phenomenon of *resonance*.

In a paper entitled "Old and New Economic Doctrines," Dr. Joseph Mayer of Washington, D. C. enumerated twenty-five presuppositions and implications of traditional classical and neo-classical logic, "together with the grounds for their repudiation if a truly scientific approach to current economic problems is to be envisaged." In suggesting a more modern point of view, he pointed out "that the modern business competition is far from being 'free,' either on the side of demand or supply" and that competition was of a "decidedly predatory nature until modern social legislation" stepped in. "Cooperative action, in setting the rules of the economic game, lifts competition to a higher plane of freer and more equitable competition." This fact and others, especially those of a psychological and institutional import, seem to vitiate the whole of the classical and neo-classical contentions. They should be kept uppermost in mind in realistic studies of modern economic problems.

A third paper, "An Interpretation and Integration of the Malthusian Hypotheses," was read by H. H. Germond of the University of Florida. A final paper "Demand for Boots and Shoes as Affected by Income and Price Levels" was presented by Victor von Szeliski and Louis J. Paradiso. This paper will appear in ECONOMETRICA in the near future.

In a general session of the American Association for the Advancement of Science Thursday afternoon, Vannevar Bush of the Massachusetts Institute of Technology spoke on "Mechanical Analysis."

On the morning of the final day of the meetings the general topic was "Financial Aspects of Building Activity," with Charles F. Roos presiding. In a paper entitled "Effects of Building Activity and Other

Factors on Security Prices," Mr. Alfred Cowles 3rd, of the Cowles Commission for Research in Economics, presented a multiple correlation of .55 between bank deposits leading stock market prices by nine months and dollar value of new building simultaneous with stock prices for the period 1886 to 1935. He explained that bank deposits represent volume of money, and building, the rate of investment, the latter a velocity factor. He pointed out that the bank deposit series showed a 40-month cycle, previously reported by Professor Davis to exist in the stock market, and that the high correlation found between bank deposits and security prices was not, therefore, surprising. In discussion Dr. William H. Newman said that building exhibited a 60-month cycle which Professor Davis had also found in the stock market, and that this probably accounted for more of the correlation.

Dr. Spurgeon Bell of the Federal Home Loan Bank Board, spoke on "The First Mortgage Market and the Bearing of the Operations of the Home Loan Board on its Recovery." Dr. Bell discussed the activities of the Home Owners Loan Corporation in taking over mortgage loans, thereby replenishing the working capital of the building and loan association and the insurance companies. He also considered the aid given by the other Federal agencies in the farm loan mortgage field and stated that the volume of new mortgage financing should be materially larger in 1936.

At the final session Friday afternoon, Mr. Roy Wenzlick of Real Estate Analysts, Inc., discussed "The Relation of Farm to Urban Values," pointing out that the general movements were similar but out of phase. Dr. Charles F. Roos of the Cowles Commission for Research in Economics presented a paper entitled "Effects of Credit, Building Costs, and Rent, on Building Activity." Dr. Roos, following up a study made three years ago by himself, Victor S. von Szeliski, and Roy Wenzlick, combined rent, occupancy, taxes, replacement costs, foreclosures, and building need (a quantity calculated from the number of families), into an integral formula correlating .93 with actual new building over a period of 40 years. He said that the quantity, rent × occupancy − taxes, represented a base value of property which was modified by a factor largely representing a psychological discount variable and also market supply of existing property and involving foreclosure rates. He explained further that builders compared this composite value factor with construction cost and actually began construction when the ratio was sufficiently favorable, that is, when it was greater than unity. He also used the foreclosure rate to measure the availability of loans, a purchasing power factor, the former leading the latter by two years.

In the final paper Dr. William H. Newman of the James C. McKin-

ney Company, Chicago, spoke on "Fluctuations in Private Building Activity and Business Cycle Theory." He discussed the relationships between the fluctuations in private building and certain economic factors commonly regarded as the causes of business cycles. Long time major cycles and shorter minor cycles were quite evident in the building index. He concluded that, (1) annual changes in population correspond closely with the major cycles and appear to be the underlying cause of these cycles, (2) changes in interest rates on real estate loans seem to have little effect but the availability of funds seems to be the most important factor in the minor cycles, (3) uncertainty in forecasting future demand plays an important part in determining the amplitude of the fluctuation, and (4) very little connection was found between building costs and building cycles. In a lively discussion, Dr. Roos pointed out that Mr. Newman's conclusion (4) concerning cost did not necessarily contradict his own theory given in the previous paper, as it was not cost directly but the ratio of present value of existing property to cost that was important in determining building activity; while Mr. Newman's approach on other factors was different, it led to similar conclusions, for instance, (1) changes in population used by Newman corresponding roughly to his own building need, (2) foreclosure rates measure the availability of funds two years later, and (3) the lag introduces over-building in prosperity periods and under-building in depressions.

HERBERT E. JONES

Cowles Commission for Research in Economics
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NOTICE OF THE MEETING OF THE ECONOMETRIC SOCIETY IN OXFORD, ENGLAND, SEPTEMBER 1936

The Sixth European Meeting of the Econometric Society will be held at New College, Oxford, England, during the weekend 25–29th September 1936. Accommodation with full board will be reserved in New College from Friday evening (dinner) at a charge of 12 sh. 6 d. per day, inclusive of all gratuities. The return fare from London is 10 sh. 9 d. Members desiring to attend the meeting are requested to write at an early date to Mr. E. H. Phelps Brown, New College, Oxford, enclosing a remittance of 2 sh. 6d. They will receive in reply further details regarding programme, accommodation, etc.

Drafts or summaries of papers should be sent before 1st July to Mr. J. Marschak, All Souls College, Oxford. An attempt will be made this year to discuss surveys on recent developments within econometrics, including economic theory. This will be in addition to the regular papers.

ERRATA

Owing to the fact that it was impossible for the author to correct galley, the following errata must be reported in the article, "Annual Survey of General Economic Theory: The Problem of Index Numbers," by Ragnar Frisch, ECONOMETRICA, January 1936, Vol. 4, No. 1, pp. 1–38.

Formula (4.9) In the summation read: $\sum_k \pi_t^k q^k \cdot \check{\pi}_t^{kh}$

p. 15 line 10 from bottom, instead of "independent of the path" read: "constant along the path."

Formula (5.3), instead of $Ic(q_0)$ read $I(q_0)$

Formula (5.7), instead of $0(q)$ read $I(q)$

p. 19 line 4, instead of q read q_*

p. 22 line 7 from bottom, instead of "psychic" read "physical"

Formula (5.18), instead of π_0 read ω_0

Formula (5.21), instead of 0_1 read I_1

Second line after (5.21), instead of $R(q_0)$ read $R_0(q_0)$

Formula (5.24), omit the superscript P

Formula (5.28), in the last equation, instead of q_0' read q_0''

Formula (6.8), the first parenthesis should read $(\omega_1\rho_1 - \omega_0\rho_0)$

p. 29 line 6 from bottom, instead of (6.14) read (6.12)

Formula (6.16), instead of V read D

p. 32 line 18, instead of "P_{0t}^{Ind} is always" read "P_{00}^{Ind} is always"

Formula (7.9), instead of $P_t =$ read $\check{P}_t =$

Formula (7.15), instead of $(\check{w}r)$ read $\check{w}(r)$

Formula (7.16), add a plus sign in the first numerator and after the first fraction

p. 35 line 15, instead of $\partial\overline{U}/x^{i}$ read $\partial U/\partial x^h$

Formula (7.24), add a plus sign after the fraction

p. 37 line 11, read "$\check{\omega}_0$ and $\check{\omega}_2$ are observable. Indeed, $\check{\omega}_0$ is equal to $\check{w}_0$"

p. 37 line 16, instead of P_0 read P_2

MONOPOLISTIC COMPETITION AND THE HOMOGENEITY OF THE MARKET[1]

By F. Zeuthen

I. INTRODUCTION

There is still a discussion going on about conditions in markets with competition among a small number of sellers. A multitude of different viewpoints and methods of distinction are being discussed and most of them will, in the end, have to find a greater or smaller place in a final system. It is most likely, however, that a prolific chaos will prevail for a number of years still to come.

Markets may be divided according to the number of sellers and buyers, conditions of demand, conditions of cost, homogeneity of the position of individual sellers and buyers, and institutional and mental conditions. The latter form a starting point for the price policy and the formal behavior of the parties, i.e., they partly determine which party mentions the price and which the quantity or, perhaps, both figures, leaving the other party to accept or refuse. They also contribute strongly to the determination of whether the parties, or some of them, take the behavior of the competitor as given data or try to influence it. In the following article, which is meant as a contribution to the preparatory work, I am only going to deal with one particular method of distinction between markets, i.e., according to their *homogeneity*. The homogeneous market mentioned below corresponds to the ordinary strict market concept,[2] the single-price market and the multiple-price market mentioned on page 197. The other types of market or spheres mentioned go more or less beyond the limit of the ordinary narrow concept of a market.

There has been a tendency in the last few years to abandon the conception of a homogeneous market with one price for all sellers and an aggregate demand function. I think the assumption of non-homogeneous markets is generally the most fruitful.[3] It seems, however, to be a

[1] I am very grateful to Dozent Dr. E. Schneider of Bonn, for his critical revision of my manuscripts.

[2] Ohlin, *Interregional and International Trade*, p. 5.

[3] Local differentiation of the market is discussed by Hotelling, "Stability of Competition," *Economic Journal*, 1929; E. Schneider, *Schmollers Jahrbuch*, 1934, and Econometrica, 1935; Zeuthen, *Quarterly Journal*, 1933, and more completely in Danish with an English résumé in *Nationaløkonomisk Tidsskrift*, 1933. (To Schneider, who has up to now penetrated farthest into the problem of local differentiation, my only objection is that his proof fails when he asserts that the price question is indeterminate in some cases of Polypoly without price discrimination, fahrb. p. 271–272. With the given assumptions, P_1 has no means

question whether the conception of markets with some degree of homogeneity and, consequently, some sort of aggregate demand functions, is not in certain cases a useful tool.

In order to bridge over the difficulties with regard to the other also very important distinctions between the markets, we shall abstract completely from price differentiation. Further, in section IV D, in connection with our problem of homogeneity, we shall consider, but only briefly, the important problem to what extent a seller is taking the sales of his competitors as given (autonomous or polypolistic price policy), or to what extent he wants to influence it (conjectural, superior, or hyperpolistic price policy).[4] In the latter case he must base his actions on his own assumptions as to the reactions of the competitor (conjectural functions; cf. also stockastical price policy). In the rest of this article we assume, for the sake of simplicity, an autonomous behavior on the part of the sellers, i.e., that they do not consider the reactions of competitors but take their sales as given reductions of the aggregate demand of the market. Very probably much the same can be said about the effects of the homogeneity or heterogeneity of the market under another assumption as to the behavior of the firms (e.g., hyperpolistic policy). Finally, the very important possibilities of co-operation or dynamic fight will not be considered in this article.

II. TYPES OF MARKETS

1. *Homogeneous Markets.*—Individual sellers are not alike, any more than individual buyers. Also, price, quality, and conditions, are not always uniform in what is generally considered a "market." In a homogeneous market exactly the same quality of a commodity is sold at exactly the same price and on exactly the same conditions in all cases.

In order to get a basis for statistics or for a more than casual reasoning, it is necessary in practice to deal with non-homogeneous markets, but as an approximation to treat them as if they were homogeneous.

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of breaking the connection between P_2 and V_1. To do that, a dynamic fight or hyperpolistic behavior is necessary.) Differentiation especially as to quality is discussed in Chamberlin, *Monpolistic Competition*, Chapters IV, V, and Appendix A. More general theories about the non-homogeneous market are to be found in Frisch's article, "Monopole-Polypole—La notion de la force dans l'Économie", *Nationaløkonomisk Tidsskrift*, 1933, and in his more detailed Norwegian lectures; and in Divisia's lecture at Leyden, ECONOMETRICA, 1934, p. 197. See also my book *Problems of Monopoly and Economic Warfare*, Chapter II, where different individual demand functions are discussed, especially in case of small temporary price differences (cf. 3 and 4 below).

[4] Cf. the articles by Frisch and Schneider mentioned above, and Kurt Sting, 'Die polypolische Preisbildung," *Jahrbücher für Nationalökonomie*, 1931.

Furthermore, the idea of a homogeneous market is a very useful tool in economic theory if one keeps its shortcomings in mind. A sub-group of No. 1 includes cases with an absolutely or approximately uniform price in accordance with costs (where costs are equal), but with unequal sales. All individual elasticities are here infinite, or approximately infinite.

2. In *non-homogeneous markets* there may be differences in the prices of firms even in the long run. In some cases one price formally prevails in a market, but the same price is taken for commodities with an unequal amount of secondary usefulness, i.e., different prices are paid for the same abstract good. The cause of the variation in price may be local distance, or difference in the quality of the goods. That the whole group still, from an essential point of view, belongs to the same market or sphere, is seen from the close connection between prices and quantities sold. When, as in the present article, differences in the quality of the goods or supplementary services are considered as differences in price, it is probably almost impossible for firms not to compete by means of price changes. The variety of the forms of insurance policies, for instance, tends very much to facilitate price competition. Even firms bound by agreement can scarcely abolish indirect price competition in the form of services, etc.

3. A special kind of market between the homogeneous and the non-homogeneous is the *temporarily non-homogeneous market*. In this market differences between the prices of the individual firms may prevail for a time, but then a process begins to take place. Customers are gradually transferred; the greater the difference in price, the more quickly the transference goes on. In case of small differences in price or, more often, in conditions or quality, an alteration up or down in the price of one firm may have very small or no effect on the sales of competitors. Cf. Case 4. But, if we assume that the market is only temporarily non-homogeneous, no firm can, in the long run, take a higher price than the others; all firms may, however, as we shall demonstrate, have a profit.

4. Another possible type of market is the *approximately homogeneous market*. Small differences in price (very likely differences in conditions or quality) may prevail here permanently. The differences in price, however, are too small to be measured and explained in the same way as in the cases where distance and costs of transportation are decisive (Case 2). The essential point is that a limited elasticity of individual demand in the case of small differences in price may bring about an equilibrium above costs, but, at the same time, a high elasticity in case of greater differences in price will insure an approximately common price in accordance with an aggregate demand function. The distinc-

tion between non-homogeneous and approximately homogeneous markets is only one of degree. The practical distinction is: In which cases is an analysis of the individual prices of the firms possible and fruitful, and in which cases is the inexact conception of an aggregate demand function a practical tool? When prices are approximately equal, any (nearly uniform) level of individual prices means a definite aggregate sale.

Cases 3 and 4 deal with different possible assumptions for inelastic individual demand curves (short time and small changes). In section IV F, another line of distinction applicable to both cases is mentioned, a distinction according to the degree of connection between individual sellers and buyers.

5. This case concerns a *non-homogeneous market with similar conditions for all sellers* and, therefore, one price or approximately one price.[5] Groups of customers are here so strongly bound to the individual firms that greater price differences are necessary to set them in motion. The price is uniform, however, or approximately uniform, for all firms, since conditions are similar for them all. Their sales may differ in accordance with the size of their special markets. This is rather a special sub-division of Case 2 due to coincidence of individual conditions.

III. PRICING IN THE FIVE TYPES OF MARKETS

ad 1. The discussion about the Polypoly theory of Cournot[6] seems to indicate that, with the assumptions of an absolutely *homogeneous*

[5] Cf. Joan Robinson and Pigou, *Economic Journal*, 1932–33, and Joan Robinson, *Imperfect Competition*, 1933, Chapter VII.

[6] A very illuminating statement of the discussion about the old Cournot problem is to be found in Chamberlin, *The Theory of Monopolistic Competition*, Appendix A, even though the reader may not agree with the conclusions of the author. A. J. Nichol in the *Journal of Political Economy*, February 1934, (cf. his article in the *Quarterly Journal of Economics*, Feb. 1934) says that Cournot's solution is only correct in the rare cases when the buyers mention the price, and the sellers merely adjust their prices accordingly. Moreover, he attacks me (page 92) as an unfaithful interpreter of Cournot. But, as quoted by himself immediately before, it is the *solution* of Cournot, i.e., his *result* and the assumptions most likely to bring it about in practice, that I am dealing with, and not Cournot's own way of reasoning. The examination of the realistic conditions necessary to bring about that or similar results made me abandon the idea of the homogeneous market as well as Cournot's own assumptions; cf. *The Theory of Monopoly and Economic Warfare*, pp 26, 29, 30, 45, 49, 50, 65–66, and 72. Only on the first of the pages mentioned, especially in the note, a short *interpretation* of Cournot is given.—See also the article by Frisch mentioned above, E. Schneider, *Reine Theorie monopolistischer Wirtschaftsformen*, and Tinbergen's Survey in ECONOMETRICA, 1934.

Since the writing of the present article, the following important contributions have appeared: Hicks' Survey in ECONOMETRICA (January 1935); Kaldor, "Mar-

market, it is only possible to reach a polypolistic equilibrium with a price in excess of costs if very rare and artificial conditions are assumed, such as sellers only varying their quantities and buyers fixing the price. The habit of applying the assumptions of homogeneity to free competition without any temporary or permanent preferences for the individual sellers and the inclination to transfer them to the problem of Polypoly have prevented many economists from understanding the actual problem of Polypoly. The important question for practical economy, therefore, is not the problem of Polypoly in homogeneous markets, but in the different forms of imperfect markets. If we have a few sellers with a polypolistic behavior in a perfectly homogeneous market, the pricing mechanism will work just as in a market with a great number of competing firms.

ad 2. Monopolistic competition in *non-homogeneous markets* is a much more practical problem. The case of a locally non-homogeneous market offers the easiest exact treatment of the problem, especially when the costs of transportation are proportionate to the distances. The influence on price policy of distance with regard to quality (which may have many dimensions) must also be considered. This is the problem of substitution and price policy.

Aggregate supply and demand functions for all sellers and buyers must here be replaced by an analysis of the interplay between the individual buyers and sellers. No aggregate demand function for the total market with one price and one quantity will be able to indicate the effect of the total demand, neither will a function between the price in any representative or arbitrary point of the market and total quantity, nor a function between average price and total quantity. The total quantity sold is a function, not of one price, but of all the different prices in the individual parts of the market, and the price differences between them are not fixed because different combinations of actual transports will be realized in different price situations. Goods may be sold outside the representative point at a price equal to that at the point plus or minus an amount smaller than the costs of transportation.

Assuming costs of transportation in proportion to distance, only points on the straight line between two firms, one of which is selling to the other, can have prices in harmony with both of them. Sales transacted between a neighboring seller and buyer, but not passing the representative market point, are transacted at prices more favorable

ket Imperfection and Excess Capacity," *Economica* (February 1935); and Stackelberg, *Marktform und Gleickgewicht;* cf. my review of it in *Geitschrift für Nationalökonomie.*

to both than if the transport had to pass the point. In a market with a multitude of prices, it is only possible to apply one representative price and an aggregate demand function in the very special cases when all traffic between sellers and buyers passes through or moves either to or from one point in a linear or star-shaped system (the multiple price market, which is a market in the strict and narrower sense of the word).

The study of the possible equilibria, or possible routes of evolution under the assumptions of non-uniform markets, is probably the most fruitful field within the whole theory of monopolistic competition. The exact method in this case is to consider the position of all individual sellers and buyers and the distances between them. In some cases, however, it seems more useful as an approximation to apply the fiction of an aggregate demand function.

Instead of considering a certain number of firms within a certain sphere, we may choose another starting point for the study of price policy in non-homogeneous markets, the consideration of the individual firm in its interplay with other firms. We may then ask: How big is the market of that firm; how far removed (locally or as to the quality of the goods) are its direct and indirect competitors? How small and far removed forces is it practical to take into consideration? This formulation of the problem means the application of Business Economics as a method of Social Economy.

ad 3–5. In all these cases there exists an aggregate demand function for the whole market and individual demand functions.

In Case 3 the temporary heterogeneity makes a monopolistic price policy possible; it protects the firms which begin an upward price movement in the direction of the polypolistic equilibrium and prevents them from abandoning it. On the other hand, the long-run homogeneity leads to a uniform price on the basis of the long-run aggregate demand curve. At any given moment a short-run equilibrium prevails with certain price differences between the sellers; but it is a dynamic equilibrium containing certain rates of change. When the static equilibrium price has been obtained, the prices of all firms will be equal, corresponding to the aggregate demand function of the market. At the same time each firm will be in an individual equilibrium with its short-run demand conditions. By this distinction between two sets of demand functions, the paradox is resolved: adjustment by means of temporary individual price changes in spite of the uniformity of price in the long run. Adjustment by means of raising the price is as possible as adjustment by price reductions, but it may take place by slow degrees if the initiating firm is not sure that the other firms want to follow suit.

In Case 4, where the individual demand functions are inelastic in case of small differences in price and conditions but very elastic in case of greater differences (cf. Figure 2D and Figure 5A), prices above costs may be realized, but at the same time an approximately common price has to prevail.

In the less important Case 5, the uniform price depends on a coincidence of the conditions of the firms.

IV. ANALYSIS OF MARKETS WITH ONE PRICE ABOVE COSTS (CASES 3 AND 4)

In the present section we shall try to sketch out some hypothetical constructions combining prices above costs (without any co-operation between the firms) and uniform or approximately uniform prices for all firms. In the first part (subsections A–C) we assume, without further explanation as to the nature of the demand functions, individual short-run demand functions which rule the price policy of the firms and an aggregate demand function determining long run equilibrium (Case 3). The nature of the individual short-run demand functions is discussed in section D. Generally the short-run individual demand functions dealt with in the first part (subsection A–B) can, without much modification, be replaced by inelastic individual demand functions for small changes in price (Case 4). The special conditions of this case are described in subsection E. The general assumption of section IV is that the difference between short- and long-run demand functions and between small and great differences in price (Cases 3 and 4) have similar effects and, furthermore, that there is a certain co-operation between the determining forces of the two cases. In section F we discuss another line of distinction, the distinction according to the existence of permanent preferences, even at a common price, for individual sellers in groups of buyers.

A. Individual and aggregate demand functions; the power of extension. —Only in markets where competition is unrestricted and where there exist no preferences are all sellers likely to have the same individual demand functions (Case 1). Otherwise, location, quality of produce, reputation, personal charm, and number and size of shops or retail connections, cause sales to differ at the same price, and the same price changes to have different effects for each firm. This generally results in a multitude of prices (Case 2). In the cases considered here, however, an approximately uniform price prevails in the long run.

The relation between the long-run aggregate demand function and the individual demand functions determining the momentary price policy of the firm is very important. The individual short-run demand function is not one continuous curve. But, assuming that the price is

approximately the same for all competitors, we have for each approximately common price and each corresponding point on the aggregate demand curve an angle or gradient indicating changes in the sales of the firm in case of small price changes and deviations from the price of the competitors. When the firm is in equilibrium, the conditions of absolute monopoly in relation to its individual demand have to be fulfilled.

Now the said gradient $dq_a:dp_a$ may be either greater or smaller than the gradient of the aggregate curve $dq:dp$. If we go back to the homogeneous market with unrestricted competition (Case 1), the individual gradients are level ($dq_a:dp_a=\infty$). The other marginal case is a "market" which is really the sum of a number of independent markets (here $dq_a:dp_a+dq_b:dp_b \cdots = dq:dp$). The market dealt with in the present section is to be found between these two limits:

$$\infty > dq_a:dp_a + dq_b:dp_b \cdots > dq:dp.$$

A change of price on the part of one seller has different effects on the transfer of customers to or from the competitors and to and from the unsatisfied demand. In Figure 1 we have two sellers A and B; the transfer between A and B, in case of a small change in A's price, is $dq_{ab}:dp_a$, and between A and the unsatisfied demand $dq_{ao}:dp_a$, indicated in the figure by the angles having the tangents $q_{ab}:p$ and $q_{a0}:p$. In the figure, we assume constant and common costs CC'. DD', the aggregate demand curve, is a straight line. At the point of equilibrium illustrated in the figure, the price in excess of costs is p, and the firms sell q_a and q_b; $q_a=q_{ab}+q_{a0}$ and $q_b=q_{ba}+q_{b0}$. The elasticity of the individual demand $\dfrac{dq_a}{q_a}:\dfrac{dp_a}{p}=1$; consequently $\dfrac{q_a}{p}=\dfrac{dq_a}{dp_a}\cdot$

Probably, in a not absolutely homogeneous market, we have $q_{a0}+q_{b0}>q_0>\begin{cases} q_{a0} \\ q_{b0} \end{cases}>0$. The gradient of the *individual demand* of a firm, $dq_a:dp_a=dq_{ab}:dp_a+dq_{a0}:dp_a$, may be called its *power of extension*, the total power being the sum of the firm's power of extension towards competitors and towards the unsatisfied demand.[7] In an absolutely homogeneous market, we have $dq_{ab}:dp_a$ and $dq_{ba}:dp_b=\infty$, and the price in excess of costs, $p=0$. If the methods applied here should give the same results as Cournot obtains by his method, we should have $dq_a:dp_a=dq_b:dp_b=dq_0:dp_{ab}$ (the gradient of all individual demand curves equal to that of the aggregate demand curve), which may either be obtained from $dq_{a0}:dp_a=dq_{b0}:dp_b=dq_0:dp_{ab}$ and $dq_{ab}:dp_a=dq_{ba}:dp_b=0$

[7] In my book on monopoly, I measure the power of extension taking the gradient of the aggregate demand as the unit.

or by coincidence from an equal opposite deviation in the power of extension towards the competitor and towards the unsatisfied demand. Both situations seem very unlikely, but not absolutely impossible. Probably the power of extension is in most cases greater and the price in excess of costs smaller. The assumption of a limited power of extension can give a series of prices more or less above costs. Simultaneously,

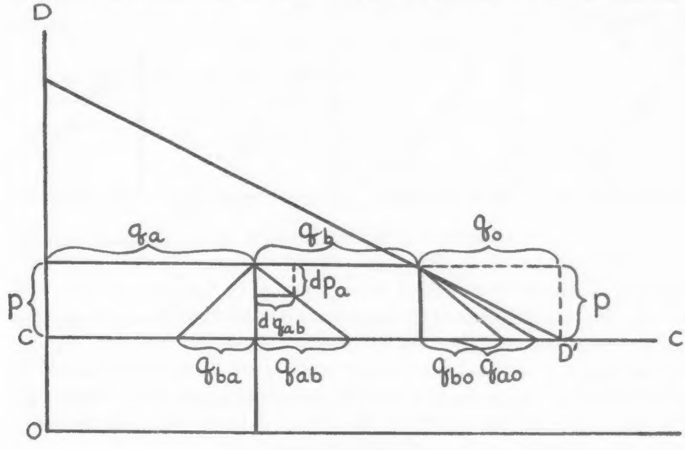

FIGURE 1

the difference between the gradients of the individual demand, the different powers of extension, leads to the well-known difference between the size of the firms even if costs are equal. The possibility of a price in excess of costs in spite of a number of competitors, and not the exact height of this price or the volume of sales, is the very valuable new idea which was first described by Cournot, but which in his own case was connected with certain special, rather unrealistic conditions.

Generally, it is impossible in a figure, besides a curve for the aggregate demand, to draw individual demand curves for each seller. Corresponding to each price set by a competitor we have separate individual demand curves for the other sellers. Since, in the case of greater price differences, the seller who varies his price will very soon destroy his competitors or himself be destroyed, it suffices to attach to each point of the aggregate demand curve indications of the gradient of the individual curve, i.e., to indicate the power of extension as an angle from a vertical line. The power of extension of a firm may be greater (as in Figure 2 A, see the marked angles $GFE > HFE$) or smaller (as in Figure 2 B) than the gradient of the aggregate demand curve.

If it is equal to it—as corresponding to Cournot's result—we may call it 1. It may vary as in Figure 2 C, or be constant as in Figures 2 A and B. If we do not consider very small price differences, we may for each price set by B have demand curves for A, for instance, as indicated in Figure 2 D. If there are three sellers, we have, when considering A, to draw a separate curve for each combination of the prices of B and C.

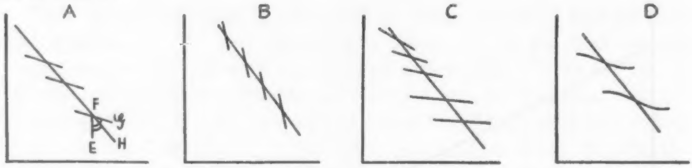

FIGURE 2

B. Formulae for the equilibrium.—We shall first consider the simplified case in which the power of extension of each firm $g_a = dq_a : dp_a$, and $g_b = dq_b : dp_b$, the gradient of the aggregate demand curve $g_{a+b} = dq_0 : dp_{ab}$ and the common average costs are all given data and constant. Applying the assumptions and nomenclature of Figure 1, including the calculation of prices in excess of costs (or the assumption of no costs), we find that the maximum profit of the firm A is to be found at the point where the elasticity of its individual demand is 1.

$$\frac{dq_a}{dp_a} : \frac{q_a}{p} = g_a : \frac{q_a}{p} = 1. \tag{1}$$

For B we have, correspondingly,

$$\frac{dq_b}{dp_b} : \frac{q_b}{p} = g_b : \frac{q_b}{p} = 1. \tag{2}$$

The aggregate demand functions and the given constant costs may, (cf. Figure 1) in order not to introduce more symbols than necessary, be expressed by the given gradient of the demand curve,

$$\frac{dq_0}{dp} = g_{a+b} = \frac{q_0}{p}, \tag{3}$$

and the sales at the point of intersection between the cost and demand curves,

$$q_a + q_b + q_0 = k. \tag{4}$$

These four equations with the given quantities g_a, g_b, g_{a+b}, and k, enable us to find the price, p, and the quantities sold and unsold, q_a, q_b, and q_0, at a price equal to costs.

If we had considered the power of extension towards the competitor and the unsatisfied demand separately, we should have had four more

equations (for g_{ab}, g_{a0}, g_{ba}, and g_{b0}) and four more unknown quantities.

If, in Figure 1, $DC=1$ and $CD'=1$, Cournot's result is obtained by inserting in equations (1) and (2),

$$g_a = dq_a\!:\!dp_a = q_a\!:\!p = 1, \quad g_b = dq_b\!:\!dp_b = q_b\!:\!p = 1,$$

and, for simplicity, substituting for (3) and (4),

$$q_a + q_b + p = 1,$$

giving

$$p = \tfrac{1}{3}; \quad q_a = q_b = \tfrac{1}{3}.$$

It might have been the result of

$$\begin{aligned} &dq_{a0}\!:\!dp_a = dq_{b0}\!:\!dp_b = 1, \quad \text{and} \\ &dq_{ab}\!:\!dp_a = dq_{ba}\!:\!dp_b = 0, \quad \text{or} \\ &dq_{a0}\!:\!dp_a + \alpha = dq_{b0}\!:\!dp_b + \beta = 1, \quad \text{and} \\ &dq_{ab}\!:\!dp_a - \alpha = dq_{b0}\!:\!dp_b - \beta = 0. \end{aligned}$$

Simple examples of an unequal power of extension are:

$$g_a = q_a\!:\!p = 2, \quad g_b = q_b\!:\!p = 1,$$

$$q_a + q_b + p = 1,$$

giving:

$$p = \tfrac{1}{4}, \quad q_a = \tfrac{1}{2}, \quad q_b = \tfrac{1}{4},$$

and

$$g_a = \tfrac{1}{2}, \quad g_b = 1,$$

$$q_a + q_b + p = 1,$$

giving

$$p = \tfrac{2}{5}, \quad q_a = \tfrac{1}{5}, \quad q_b = \tfrac{2}{5}.$$

We now abandon the assumptions of a constant power of extension and a constant gradient for the aggregate demand curve and substitute other functions of the unknown quantities (p, q_a, and q_b). We may, e.g., assume constant elasticities in case of individual price changes, or let the power of extension depend on the relation between the sales of the firm and those of its competitor; most likely it is a function of the two or more quantities sold. If the functions are of a higher degree, there may be several solutions—as may be the case in other economic fields—but a limited number of determined solutions. Special problems exist where one has a numerical inequality between $dq_a\!:\!dp_a$ in case of reductions and increases in price, and other problems where there are systems of individual demand curves with more than one maximum, i.e., where it is doubtful whether greater or smaller price changes have to be considered.

Neither will different and changing cost functions for the firms disturb the determinateness of the solution. In Figure 3 the solution is given for two firms with different average cost curves, C_aC_a' and C_bC_b', and a constant power of extension with the gradients $(dq_{a0}+dq_{ab})\!:\!dp_a=g_a$ and $(dq_{b0}+dq_{ba})\!:\!dp_b=g_b$. The figure to the right illustrates the polypolistic curve of reaction for A, C_aM_a, by com-

bining the cost curve with the constant power of extension, i.e., the tangents to the different possible individual demand curves corresponding to the different quantities sold by the competitor. The series of equilibrium points obtained form the reaction curve. From this supply curve and a similar curve for B, C_bM_b, we find the aggregate total curve C_aXM_{a+b} and, by intersection with the aggregate demand curve DD', the price p and the quantities sold by the two firms, q_a and q_b. A's profit is $BEGL$ and B's $BFHK$.

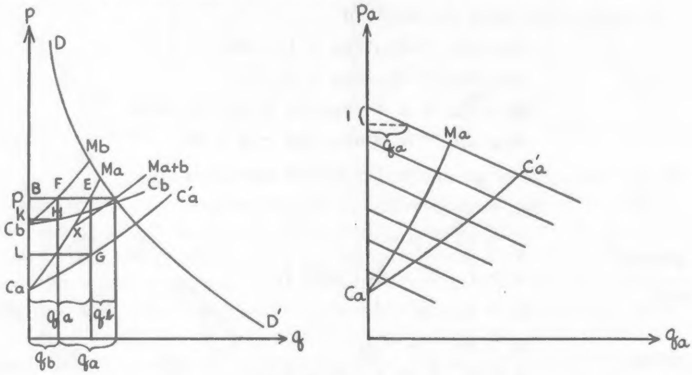

FIGURE 3

If the power of extension for A, measured as an angle, had not been constant, but a function of p_a and q_a, we should have had in the figure to the right a system of determined but more complicated lines, hyperbolae, e.g., if the elasticity was constant. If the power of extension, i.e., the variation in individual demand, also depends on other of the quantities mentioned, the simple graphical method fails but the equations then show that the solution is still determinate. In real life the correlation with q_b will be very great.

In the equations on page 202, the substitution of different and changing cost functions for the common and constant function involves a modification of the last equations.

The sales of the individual firm depend on a combination of its own power of extension with those of the other firms, with the aggregate demand curve, its own and its competitors' cost levels, and the gradients of the cost curves.

C. The path towards equilibrium.—On the path towards equilibrium, short-run individual demand curves (or powers of extension) are of importance. In the short run, the power of extension is probably much smaller than in the long run. It is probable that in the very short run

especially, in case of small changes in price the power of extension towards the customers of the competitors is very small.

At any price above the final equilibrium, the power of extension gradually increases as time elapses, after a decrease in price. Most likely, even before they reach their full long-run sizes, disharmony arises, since the sum of all powers of extension ($\Sigma dq_i{:}dp$) multiplied by p is greater than the total possible sales at that price according to the aggregate demand curve. Some or all firms are then above their

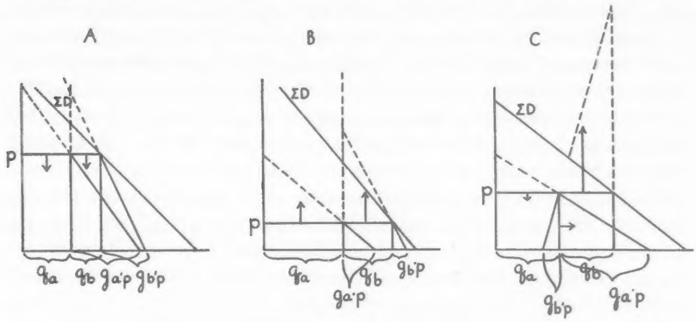

FIGURE 4

equilibrium price and want a price reduction (cf. Figure 4 A). The length of the vertical arrows corresponds to the desired price changes, so long as the sales of the competitor are as indicated. Before the equilibrium price is reached, the distribution of sales will be more or less casual. Probably the expectations of the individual long-run demand functions will start the downward movement even before the aggregate sales are too small to harmonize with the momentary demand functions (which are determined by the time elapsed since the last changes and the effects of earlier conditions).

If the price is below the long-run equilibrium, it is probably still more below the short-run or momentary equilibrium. An upward movement will then take place, and individual increases in price are possible when the transfer of customers between the firms is relatively small in case of smaller and short lasting price differences (cf. Figure 4 B). The individual firms are able to increase or decrease their prices slightly without any sudden and great changes in the sales of their competitors. If the customers do not have the same connection with all firms, the special customers of a firm will extend or contract their sales a little before the change in price gives rise to a transfer of customers from the competitors. In other words, in this way a temporary adjustment takes place by small changes in price as if it were changes in the quantity of

sales of the individual firm, leaving the sales of the other firms temporarily unaffected. The possibility of this price policy is still greater if a small amount of surreptitious discrimination is also applied. The next step is for other firms, if they find it to their interest, to make a movement in the same direction before the price differences give rise to any transfer of customers.

If the firms, however, have opposite interests in the price movement, the sales of a firm working for a higher price will decrease while those of a firm working for a lower price will expand, by a transfer of customers. This double movement in prices and quantities will continue until both are in equilibrium. In the case illustrated in Figure 4 C, one firm, A, wants lower prices, and the other firm, B, higher prices. The solution is a transfer of customers continuing until they are satisfied at the same price, which will generally deviate from the one at which they started. The horizontal arrow indicates the tendency of the movement for the frontier between the sales of A and B. At the starting point we have $g_a.p=(dq_a:dp_a)p>q_a$ and $g_b.p=(dq_b:dp_b)p<q_b$; at the point of equilibrium both those inequalities have to be transformed into equations, and the conditions of demand and costs have simultaneously to be fulfilled.

As in other fields, events on the way towards equilibrium may change the fundamental facts and thus modify the equilibrium itself.[8] Particularly, the distribution of sales while the process is going on will influence the final result. This may, furthermore, lead to a conscious dynamic fight, which is, however, beyond the assumptions dealt with in this article.

D. The individual demand function and the initiative.—At the point of equilibrium, there must be no expectation of advantage for any of the parties through any price movement upwards or downwards. The determining short-time functions or curves at the point of equilibrium refer to the expected periods or combinations of periods determining the actions of individual firms. The equality, or approximate equality, of price for all firms corresponds to an unlimited or very great power of extension towards the competitors in the very long run. The smaller power or elasticity illustrated by the individual determining demand functions refers to the *most probable periods of deviation.* They do not concern real demand reactions, but *expected* or, according to Frisch,[9] *conjectural* demand functions.

In a market which is only temporarily heterogeneous, the short-run demand function determining the price policy will most likely be deter-

[8] Kaldor, "A Classificatory Note on the Determinateness of Equilibrium," *Review of Economic Studies,* 1934.

[9] Cf. note 4.

mined by the quantity sold by the initiating firm after the competitors have had time to follow the price movement and have adjusted their sales accordingly (hyperpolistic adaptation); here, too, the decisive factor is the sales that the firm expects to obtain (conjectural functions). That means that the polypolistic assumptions hitherto presupposed are very likely in many cases to be replaced by more complicated assumptions as to the behavior of the parties. Two sellers, alone in a market for years, will probably try to influence each other's tactics. A seller controlling, e.g., 20 to 30 per cent of the sales in an unsettled market with several changing competitors, is more likely, in accordance with the simple polypolistic assumptions, to consider the sales of his competitors only as an unalterable reduction of the aggregate demand of the market because the determining conditions are beyond his knowledge and control. The determining short-run demand functions will in any case be based on a conjecture as to the reactions of the customer, possibly also, more or less, as to the reactions of the competitors. The long-run demand functions, however, indicate real reactions of prices and sales.

The great lasting effects of one seller alone changing his price are excluded or limited by the temporary heterogeneity. There may still remain in some cases in the long run an effect of the initiative—favorable to the initiating firm in case of price reductions and unfavorable in the opposite case. When a seller is in equilibrium, he will find any price movement initiated by him disadvantageous.

E. Limited individual elasticity for small differences in price (even in the long run; Case 4).—Price-determining individual demand curves with limited elasticity are not only the result of the short-time immobility of consumers. They may also appear if the elasticity is small in case of small individual differences in price (or in quality, conditions, etc.), but great in case of greater differences. The inelasticity is most likely to appear in cases of simultaneous small and short differences, but here we shall consider a case where the demand functions are independent of the time of adaptation. At the same time *another new condition* of demand is introduced, i.e., *preferences* of certain groups of buyers for individual sellers, also when their prices are, and have been, the same for a long time.

In Figure 5 A, the line C, $A+B$ indicates the aggregate demand of the two firms A and B *when* both fix the *same price*, or approximately when the price is nearly the same. CA and CB' are the corresponding individual demand curves, the latter being indicated in the left part of the figure where B's sale is placed nearest the axis. If, however, the prices are essentially different, e.g., differ by more than 5 or 10 per cent, the aggregate demand curve C, $A+B$ will be the demand curve for the

firm with the lowest price, while the sales of the other firms will be reduced to zero. The dotted line through P indicates the individual demand of A, *when B's price remains* at P, and the dotted line through S gives a corresponding demand curve for B when A's price remains at S. If, now, the dotted lines are tangential to CA and CB', each firm

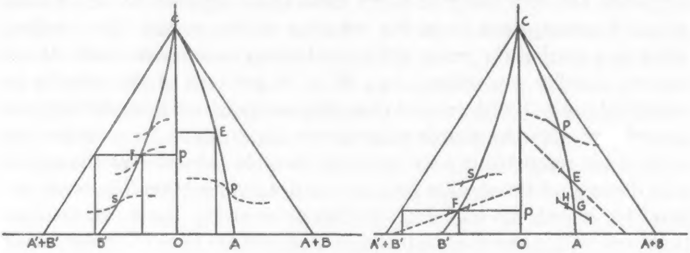

FIGURE 5

will have its monopoly point where the elasticity of the curve is 1, e.g., A at E and B at F. The lowest of these individual prices (here F) will be realized if the other firm does not prefer a still lower price, or prefers to compel the former by a dynamic fight, or the individual adaptation is replaced by some form of co-operation.

The condition in Figure 5 A of the dotted lines being tangential means that very small changes in the prices of one firm have the same effects irrespective of whether the other firms follow the price movement or not. But even if the dotted line is more elastic or, rather, as in Figure 5 B, if only price differences of small but finite sizes are relevant, the results will be similar, only at a somewhat lower price level.

In Figure 5 A and B we have not only an aggregate demand curve, but also continuous individual demand curves applying when both firms fix the same price. The situation, however, may also be as in Figure 1, i.e., only the aggregate demand curve and the individual demand functions for small price movements are given. The sales of the firms at the equilibrium price are here determined by the process described above, and all firms can be in monopolistic equilibrium at the same time. When, as in Figure 5, the market for each firm is approximately given at any price, there is no room for hyperpolistic pricing, etc.

The size of the price interval with relative inelasticity, possibly combined with its duration and other market conditions, determines whether there is a basis for a price above costs.

F. Buyer's connection with individual sellers.—As mentioned immediately above, cases with limited elasticity in regard to small changes in price (Case 4) can be of the nature illustrated in either Figure 1 or Figure 5. The same holds good in cases with temporarily limited

elasticity for the individual firms (Case 3). The lines of distinction intersect each other. In Figure 1 as well as in Figure 5, short or small differences in price may determine the individual price policy and thereby influence the price level. The difference, however, between the assumptions of the two figures is that in Figure 1 there is no *a priori* connection between individual buyers and sellers, but the distribution of sales depends on short-time (Case 3) or small-price-difference demand conditions (Case 4) in connection with the aggregate demand function. A complete equilibrium for all firms by variation in their sales is here possible. Under the assumptions of Figure 5, individual connections always exist (i.e., not only a quicker or stronger attraction *in case of differences* in price), and a distribution of sales which can be only slightly modified is thereby given. Consequently, it is only the firm with the lowest monopoly point that reaches a genuine monopolistic equilibrium. The other firm will be forced down. By taking a slightly higher price than the competitor and, consequently, by a minimal reduction of the normal sales, he may in the neighborhood of the forced price obtain the monopolistic position as to price, sales, and elasticity. In Figure 5 B, when B has fixed the price p, A will move to the top of the line G or stop at the point H if we assume the possibility of doing so. We have here actually got back to the most important Case 2, the non-homogeneous market, only that the differences in price and in the individual demand conditions are assumed to be too small to be described and measured.

V. CONCLUSION

What is generally considered a market is, as a rule, not a homogeneous sphere. The most exact method is, therefore, to consider the actual differences within the sphere and, consequently, to consider one price for each seller. As a result, we have the studies that have appeared during the last few years in price policy and locality, price policy and differentiation of produce, or about preferences for individual sellers generally. There still seems to be a sphere, however, between the homogeneous and heterogeneous markets where the conception of a common demand curve is applicable, but, on the other hand, preferences for the individual sellers have obvious effects on the distribution of sales, individual price policy, and, consequently, also on the common price of the market. Here the methods indicated above may offer a theoretical explanation and perhaps later on may be made applicable for practical use.

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AN ECONOMETRIC MODEL OF PRODUCTION AND DISTRIBUTION

By VICTOR EDELBERG

I. ANALYTICAL

1.1 I TAKE THE SIMPLE Wicksell (*Vorlesungen*, *Lectures*) class of theoretical processes of production. Professor R. Frisch calls them the "point input-point output" class.

I assume one kind of labour, one kind of land, and one kind of consumption goods.

Labour at a rate x and land at the rate l are applied in a vertically completely integrated firm at $t=0$. The "point input" is $xdt+ldt$. It produces a consumption good output emerging at $t=s$ at a rate p. The "point output" is $p\cdot dt$, and s is the period of production.

1.2 Let w be the wage rate and r the rent rate and ρ the rate of discount.

Then we have these equilibrium relations:

$$(wx + rl)e^{\rho s} = p, \tag{1}$$

and

$$\begin{aligned} k &= (wx + rl)\int_0^s e^{\rho t}dt, \\ &= \frac{p - (wx + rl)}{\rho}, \end{aligned} \tag{2}$$

= "free capital" required to finance the production process from its beginning at $t=0$ to its end at $t=s$,

= the value of the intermediate products in the firm in a position of stationary equilibrium, where identical processes are started at successive instants.

1.3 Given the entrepreneur's knowledge of alternative methods of production at $t=0$, we have the function

$$p = p(x, l, s), \tag{3}$$

which shows the largest rate of output p (which is possible under the conditions of the given knowledge) possible when using a given set x, l, s, as a function of x, l, s. I assume the function is continuous, and homogeneous of the first degree with respect to x, l but not to s, i.e.,

$$np = p(nx, nl, s),$$

where n is any number.

The competitive equilibrium condition is: of w, r, ρ, two being given, the entrepreneur adopts such a method that the third is maximised.

Using (1), supposing w, r given,

$$\rho = [\log p(x, l, s) - \log (wx + rl)]s^{-1}$$

is maximised when

$$\frac{\partial \rho}{\partial s} = \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial l} = 0$$

and the maximal value is

$$\rho = \frac{\partial p}{\partial s \cdot p}. \tag{4}$$

(4) is obtained also when in (1) ρ, r are given and w maximised, or when ρ, w are given and r is maximised. When in (1) w is maximised, the maximal value is

$$w = e^{-\rho s} \frac{\partial p}{\partial x}, \tag{5}$$

and when in (1) r is maximised, the maximal value is

$$r = e^{-\rho s} \frac{\partial p}{\partial l}. \tag{6}$$

Henceforward by ρ, w, r, I mean only the maximal or "equilibrium" values (4), (5), (6). In general, they are functions of x, l, s. I shall write c_1, c_2, c_3, c, for arbitrary constants.

On the basis of statistics which follow, I assume that the relative shares of land, labour, and capital, in the rate of output p are constant as x, l, s vary.

Hence, $p/(wx+rl)$ is constant, and using (1),

$$\rho s = \text{constant} = \alpha. \tag{7}$$

Using (4),

$$\frac{\partial p}{\partial s \cdot p} = \frac{\alpha}{s},$$

$$\therefore \quad p = c_1 s^{\alpha}. \tag{8}$$

The relative share of labour, $e^{-\alpha}(\partial p/\partial x)(x/p)$, is constant,

$$\frac{\partial p}{\partial x} \frac{x}{p} = \text{constant} = \beta, \tag{9}$$

$$\therefore \quad p = c_2 x^{\beta}. \tag{10}$$

The relative share of land, $e^{-\alpha}(\partial p/\partial l)(l/p)$, is constant,

$$\frac{\partial p}{\partial l}\frac{l}{p} = \text{constant} = \gamma, \tag{11}$$

$$\therefore \quad p = c_3 l^{\gamma}. \tag{12}$$

Combining the partial functions (8), (10), (12), we have

$$p = cs^{\alpha}x^{\beta}l^{\gamma}, \tag{13}$$

which is the empirical form of the productivity function (3) and yields constant relative shares. Since (13) is a homogeneous function of the first degree with respect to x and l

$$\beta + \gamma = 1. \tag{14}$$

Both β and γ are assumed to be positive.

I assume a theoretical economy consisting of firms identical to the firm described, except that they may differ in size. Corresponding to x, l, p, k, we have X, L, P, K, where X is the total rate of labour input in the economy at $t=0$, L is the total rate of land input in the economy at $t=0$, P is the total resulting rate of output of the consumption good in the economy at $t=s$, K is free capital for the whole economy required to finance from $t=0$ to $t=s$ the processes started by applying X, L.

Using (13), the productivity function for the economy is

$$P = cs^{\alpha}X^{\beta}L^{\gamma}, \tag{15}$$

and all the equations previously obtained can be extended from the firm to the economy by writing capital letters X, L, P, K, for the small letters x, l, p, k, where necessary; thus (2) becomes

$$K = \frac{P - (wX + rL)}{\rho}.$$

1.4 Assuming the supply of land, L, constant, there are two ways in which we can state the demand conditions for labour in the economy.

The first way is this. Using (5), (7), (15),

$$w = e^{-\alpha}\frac{\partial P}{\partial X} = e^{-\alpha}\beta cs^{\alpha}X^{\beta-1}L^{\gamma},$$

$$\therefore \quad X = \left(\frac{e^{\alpha}w}{\beta cs^{\alpha}L^{\gamma}}\right)^{\frac{1}{\beta-1}}, \tag{16}$$

and the demand elasticity with respect to the wage rate is

$$\frac{\partial X}{\partial w}\,\frac{w}{X}=\frac{1}{\beta-1};$$

using (14)

(17) $$=-\frac{1}{\gamma},$$

and is

$$<-1,$$

and the demand is elastic.

The second way is this. From (5), (6), (7), (9), (11),

(18) $$rL=\frac{\gamma}{\beta}\,wX.$$

From (7)

(19) $$\rho=\frac{\alpha}{s};$$

using (1), (18), (15),

(20) $$s=e\left[\frac{\left(\frac{\gamma}{\beta}+1\right)wX^{1-\beta}}{cL^{\gamma}}\right]^{1/\alpha}.$$

Using (18), (19), (20), I can write (2) as

$$K=\frac{(e^{\alpha}-1)\left(\frac{\gamma}{\beta}+1\right)wX}{\alpha}\,e\left[\frac{\left(\frac{\gamma}{\beta}+1\right)wX^{1-\beta}}{cL^{\gamma}}\right]^{1/\alpha},$$

$$\therefore\quad X=\left[\frac{\alpha(cL^{\gamma})^{1/\alpha}\cdot Kw^{-(1+1/\alpha)}}{e(e^{\alpha}-1)\left(\frac{\gamma}{\beta}+1\right)^{1+1/\alpha}}\right]^{\frac{1}{1+\frac{1-\beta}{\alpha}}}$$

Assuming the total supply of land in the economy is fixed and fully employed, treating L as constant, and writing

$$\eta=\frac{1}{1+\frac{1-\beta}{\alpha}}=\frac{\alpha}{\alpha+\gamma},\qquad \sigma=\frac{1+\frac{1}{\alpha}}{1+\frac{1-\beta}{\alpha}}=\frac{\alpha+1}{\alpha+\gamma},$$

and a constant

$$C = \left[\frac{\alpha(cL^{\gamma})^{1/\alpha}}{e(e^{\alpha}-1)\left(\frac{\gamma}{\beta}+1\right)^{1+1/\alpha}}\right]^{\eta},$$

$$X = CK^{\eta}w^{-\sigma}, \tag{21}$$

and the elasticity of demand with respect to the wage rate is

$$\frac{\partial X}{\partial w}\frac{w}{X} = -\sigma, \tag{22}$$

and with respect to the free capital is

$$\frac{\partial X}{\partial K}\frac{K}{X} = \eta. \tag{23}$$

The elasticities (17) and (22) are, in general, not equal, i.e., in general,

$$\frac{1}{\gamma} \neq \sigma.$$

What we mean by the "wage elasticity of demand for labour" depends on what we mean by the "conditions of the demand." (17) assumes s is kept constant and K may vary, and (22) assumes K is kept constant and s may vary.

II. STATISTICAL[1]

2.1 By the "national income" I mean the sum of (net) money payments to all the inhabitants of a country during a year, for the use of their labour and property in production. The following are statistics of distribution of the national income for the United Kingdom from 1843 to 1924.

2.2 R. Giffen gives a table showing the distribution of the national income in 1843 with a reservation that his "figures . . . make no pretence to exactness" (*Essays in Finance*, second series, p. 467).

Using his figures, we have the following table:

Distribution of the national income in 1843 in £ millions.

Income from property	190
Income of all classes from work	325
Total:	515
The proportion of the total going to work	63%

2.3 Professor Bowley (*Changes in the National Income 1880–1913*, pp. 24–6) gives the proportion of the national income "earned" as

[1] What follows owes much to the advice of Professor A. L. Bowley.

$62\frac{1}{2}$ per cent in 1880, rising to an annual average of about 65 per cent in the next twenty years, and falling again to $62\frac{1}{2}$ per cent in 1913. Professor Bowley estimates (*Economic Journal*, 1904, p. 459) that in 1860 the proportion was nearly the same as in 1913.

The word "earned" is used in the special sense employed in assessing income tax and includes not only income from work but also income derived by persons from employment of their own capital in private firms.

Professor Bowley estimates that subtracting from the "earned" category the income due to capital in private firms owned by their members, the proportion of income from work in 1913 is 61 per cent. This is to be compared to Giffen's estimate for 1843 of 63 per cent. Professor Bowley says (p. 24) that farmers' incomes, which are mainly due to work, are underestimated in Schedule B, and, therefore, the proportion due to work should be taken greater than 61 per cent. Therefore, it does not seem necessary to distinguish income from work and the earned income, which was $62\frac{1}{2}$ per cent of the national income in 1913. Giffen's estimate of 63 per cent can be taken as relating to earned income.

The approximate constancy of the earned proportion in the period from 1843 to 1913 is remarkable. Comparing 1880 with 1913, Professor Bowley writes, "though it is not proper to lay stress on the exact equality of the proportions at the two dates, yet the evidence is sufficient that any change there may have been is inconsiderable. The constancy of so many proportions found in the investigation seems to point to a fixed system of causation and has an appearance of inevitableness."

A part of the income from property is interest on capital invested abroad. Interest from abroad in 1880 was about £50 millions and in 1913 about £200 millions.

The "home produced national income" is the national income less net interest from abroad. Using Professor Bowley's figures, the earned income as a proportion of the home-produced income in 1880 is $65\frac{1}{2}$ per cent and in 1913 is 69 per cent.

The difference, $3\frac{1}{2}$ per cent, between the two proportions is not great.

2.4 Professor Bowley and Dr. Stamp exclude Southern Ireland and estimate the earned proportion for 1911 as $75\frac{1}{2}$ per cent and for 1924 as 78 per cent (*National Income in 1924*, p. 50, using Professor Bowley's estimates for 1911 published in his *Division of the Product of Industry*).

The difference between $75\frac{1}{2}$ per cent for 1911 and 69 per cent for 1913 is due to a difference in the basis of the two estimates. In estimating for 1913 Professor Bowley appears to take one quarter of the gross assessment under Sch. D as earned (*N.I. 1880–1913*, p. 24) and a

greater proportion for 1911 (*N.I. 1924*, p. 51). Southern Ireland accounts for some 4 per cent of the national income and its inclusion or omission affects the proportion negligibly.

Using the figures of *N.I. 1924*, the earned proportion of the national income in 1911 (including £194 millions interest from abroad) is 70 per cent. This is to be compared to $62\frac{1}{2}$ per cent for 1913 and the difference must be due to the change of "basis."

Conclusion. Whether we take the annual national income or its home-produced component, the earned proportion is roughly constant between 1843 and 1913 on the old (Giffen) basis, and as between 1911 and 1924 on the new Bowley-Stamp basis. The estimate of the income from work should be nearly the same as for the earned income. Hence, the proportion due to work has been roughly constant.

Mr. Clark carries analysis of the income statistics up to recent years (*National Income 1924–1931*, and *Economic Journal 1933, 4*). But they are not comparable to the Bowley-Stamp estimates. Thus, instead of the Bowley-Stamp estimate of the earned proportion $75\frac{1}{2}$ per cent for 1911, Mr. Clark gives only $54\frac{1}{2}$ per cent in Table XXV of his book. I do not go into Mr. Clark's figures because it seems difficult to recast them for purposes of comparison with the earlier estimates, especially as his methods present certain difficulties which have been pointed out in a review of his book (*Statistical Journal 1933*, p. 110).

2.5 There are three main difficulties about the income statistics.

(a) The distinction between the value of a firm's capital and its profits (on ordinary shares) is somewhat arbitrary. The firm's gross annual profits are the difference between what it pays out and what is paid to it during a year. Its net profits are gross profits less "normal capital depreciation" and net borrowing during the year. "Normal depreciation" is an arbitrary thing, it includes neither capital written off nor "capital appreciation" such as may be due to a stock exchange boom or to monetary inflation. Profits are arbitrary because normal depreciation is so. Profits are what the business world thinks they ought to be.

But, in the mass and in the long run, profits are not arbitrary. It is found that the average rate of profit on the nominal value of ordinary shares over ten or twenty years approximates to the average rate earned by debentures. The rough rule which directors follow in declaring profits is this: they declare such a rate as is compatible with maintaining profits at the above average rate in the long run.

(b) The estimate of profits is complicated by the fact that Schedule D of income tax returns does not separate income from capital in private businesses from income due to work of the owners of these businesses. Dr. Stamp (*British Incomes and Property*) goes into this,

as also R. Giffen and Professor Bowley (*loc. cit.*). A conventional basis is adopted for dividing income under Sch. D between work and property. The proportion of the national property controlled by limited companies has been constantly growing. "There has been a transference of income from the earned to the unearned category without any real change in its source" (*N.I. 1924*). Hence, the use of a fixed basis for dividing the gross assessment under Sch. D into the earned and unearned components over long periods is misleading. Unfortunately, statisticians have not been able to separate income from work and income from property in private firms. That would have solved the above difficulty. In theory, the two components are separable according to the principle of marginal productivity. With reference to the purpose to which I put the statistics, there is a third difficulty.

(c) A country is not a closed economic system. Production in the United Kingdom is only a part of the world process of production. This country imports materials and foodstuffs and exports manufactures, etc. I want information on income distribution in the whole process but only a part of it is covered by the statistics I use.

There is no reason why the relative shares of labour and property in the national incomes of different countries should be the same. Yet among industrialised countries they do not seem to differ very much. Thus, for the United States in 1910 King estimates the proportion of the national income due to work as $74\frac{1}{2}$ per cent (*The Wealth and Income of the United States*, Table XXXI). This is very close to Professor Bowley's estimate of $75\frac{1}{2}$ per cent as the earned share of the home produced income of United Kingdom in 1911.

As a rough approximation, I could perhaps assume for my purpose that the relative shares for the whole world are the same as in the home-produced income of the United Kingdom. But there is no need to assume that.

Instead, I regard this country as a *quasi* closed economy. This is done by regarding imported goods as produced in this country. They are produced by producing the export-goods which are exchanged for them. In that sense we have a closed structure of production. The total income yielded by this structure is the home-produced national income.

From Table II, *N.I. 1924* (in £ millions):	1911
The home-produced income less pensions and national debt interest:	1868
Earned:	1409

From Table II, *Division of the Product of Industry*:

Ownership of lands:	34
Ownership of buildings:	144

"Lands include farm buildings and improvements." "Buildings" include land on which they stand.

The income of land as an original factor is estimated by Sir. T. Whittaker (*Ownership and Taxation of Land*, p. 94) for 1910–11 as follows:

	£ millions
Agricultural land	21
Sites of houses and buildings	50
Land and empty properties	2
Extra value of land of underassessed properties	5
Other property including sporting rights	1
Mines, quarries, railways, canals, docks, etc.,	12.5
TOTAL	91.5

Other estimates are given in *British Incomes and Property*.

The relative share of labour has been roughly constant. Assuming the relative share of land constant (as in U.S.A., in any case it is small), the relative share of land and labour together has been roughly constant. The absolute share of labour in 1911 is £1409 millions and of land, say, £92 millions. Together they make £1501 millions. Using (15) as a model of production and distribution of the home-produced income (which is £1868 millions in 1911), putting

$$\frac{P}{wX + rL} = \frac{1868}{1501},$$

and using (1), (7),

$$\alpha = \log \frac{P}{wX + rL} = \log \frac{1868}{1501} = .219.$$

A slight inaccuracy is introduced by the fact that in (15) P is a rate of output of consumption goods, whereas about 10 per cent of the home-produced income was net capital saved, the rest being the value of consumption goods.

Using the model (15), the input rate, $wX + rL$, occurs s years (taking 1 year as unit) earlier than the output rate, P. Therefore, I should not have used above the £1501 millions estimate of the share of the original factors for 1911. I should use an estimate relating to a date s years before. From (2), s is roughly the ratio between the national capital (excluding land) and the home-produced income per annum. This ratio was about 6. Therefore, taking $s = 6$, I use an estimate of the original factor share in 1905

The national income in 1905 was between £1700 and £1800 millions. I take it as £1800, which is perhaps an overestimate, so as to compensate for the fall of purchasing power of money between 1905 and

1911. I leave out of account the growth of the proportion of the national income derived from abroad. The national income in 1911 is £2012 millions.

Then the share of the original factors in 1905 is roughly

$$\pounds\,\frac{1501 \times 1800}{2012}\text{ millions}$$

and

$$\begin{aligned}\alpha &= \log\left(\frac{\text{home-produced income in 1911}}{\text{the share of the original factors in 1905}}\right),\\ &= \log\frac{1868 \times 2012}{1501 \times 1800},\\ &= .330 = \tfrac{1}{3}\text{ approximately}.\end{aligned}$$

Using a higher value for s of 8 years (as the equation $s=\alpha/\rho$ suggests) does not alter the estimate, $\alpha=\frac{1}{3}$, appreciably, because the national income in 1903 is over £1700 millions. The real income of the original factors is greater in 1911 than in 1905 because of invention and the growth of the employed population. The effect of invention is to increase the coefficient C in (15).

The first estimate, $\alpha=.22$, is obtained by regarding the 1911 position as if it were a position of "stationary equilibrium" and the amount of the home-produced income and the absolute shares were the same year after year. This is a first approximation. Then an allowance is made for "progress" and an estimate, $\alpha=\frac{1}{3}$, is obtained. In what follows I use $\alpha=\frac{1}{3}$ as a rough estimate of α which would be obtained (using the point input-point output model) if exact measurements were available of the components of the national income.

Using statistics and (1), (5), (9),

$$\beta = \frac{wX}{wX + rL} = \frac{1409}{1501} = .932 = .93,\ \text{say};$$

using (14)

$$\gamma = .068 = .07,\ \text{say};$$

and (15) becomes

$$(15') \qquad P = cL^{.07}X^{.93}s^{.33},$$

and the elasticity (17)

$$(17') \qquad \frac{\partial X}{\partial w}\,\frac{w}{X} = -\frac{1}{\gamma} = -14.7.$$

Putting

$$\beta = \frac{1409}{1501},$$

$$\eta = .84,$$

and

$$\sigma = 3.34,$$

and (21) becomes

$$X = CK^{.84}w^{-3.3}, \tag{21'}$$

and the demand elasticities

$$\frac{\partial X}{\partial w}\frac{w}{X} = -\sigma = -3.34, \tag{22'}$$

and

$$\frac{\partial X}{\partial K}\frac{K}{X} = \eta = .84. \tag{23'}$$

The elasticity (17′) is very high and for certain reasons is not a useful concept.

But (21) is usefully applicable to many practical problems and is useful by showing clearly how the demand for labour depends simultaneously on the real wage rate and the supply of free capital. In dynamic problems, K has to be defined as the supply of free capital expected by the entrepreneurs at $t=0$ to be forthcoming in the future to finance, till their completion, the processes started at $t=0$ by the input rates, L, X, of the original factors.

Keeping K constant and allowing w to vary, (21′) becomes a model demand curve for labour in terms of the real average wage rate. The elasticity of the curve, $\sigma = -3.34$, is high. No stress is laid on the exact figure -3.34. But it seems to indicate that if experiments could be made they would show the real wage elasticity of the demand for labour to be around -3.

The result is broadly confirmed by observation. It has been observed that Trade Unions were powerless to raise the average real wage rate in this country noticeably above what it probably would have been in absence of their pressure. It seems probable that Trade Union pressure and the state regulation of minimum wage rates were together responsible for about half the post-war "permanent" unemployment of 1 million, without influencing the average real wage appreciably. The high elasticity is a disadvantage in that the average wage rate cannot be raised institutionally without causing great unemployment. It is an

advantage in that a small reduction in the average real wage rate will expand employment considerably.

The elasticity of about -3 means that (assuming K constant) a diminution of 1 per cent in the wage rate will increase employment by about 3 per cent, and a rise in the wage rate of 1 per cent will diminish employment about 3 per cent. The analysis suggests that the real income of the wage-earning class cannot be increased by an institutional raising of the real wage rate but will be diminished since the number employed would fall in a ratio about three times the ratio of the rise in the wage rate. An effective way of increasing the real income of the wage-earning class is through taxing the incomes of the rich.

That latter policy has its limits. Such taxation, especially death duties, diminishes the supply of free capital. Using the simple model (21′), it diminishes K. Since the elasticity (23′), $\eta=.84$, it diminishes employment nearly in proportion to the diminution in capital (if w is kept constant).

If an institutional rise in wages is accompanied by a simultaneous reduction in free capital, the diminution in employment may be particularly severe. Thus, if w is raised by 1 per cent and K lowered by 1 per cent, employment, X, diminishes by $\sigma+\eta$ per cent, i.e., by about 4 per cent. On the other hand, if w, K, vary in the same direction, they offset to some extent each other's effects on employment, X, as shown by (21′). But, since σ is about four times as great as η, employment is more sensitive to changes in w than in K. Therefore, the usual way of representing the demand for labour as a function of the wage rate alone turns out to be not such a bad approximation after all.

It may be noted that as w is varied in (21) and K remains constant, s and ρ will vary. Using (2) etc., $\partial s/\partial w>0$ and $\partial\rho/\partial w<0$, since $\sigma>1$ and $\partial X/\partial w=-\sigma CK\ w^{-(\sigma+1)}$ is numerically greater than $X/w=CK^{\eta}w^{-(\sigma+1)}$. The change in the rate of interest may induce a change in the supply of free capital, K. An institutional rise in the wage rate, *e.g.*, will lower the rate of interest and may influence the supply of free capital.

2.7 King (*Wealth and Income of the United States*) gives statistics of distribution of the national income in U.S.A. from 1850 to 1910. Using his Table XXXI, we have the following table of relative shares as percentages of the national income.

	Labour	Land	Capital
1850	79.8	7.7	12.5
60	76.5	8.8	14.7
70	80.2	6.9	12.9
80	72.8	8.7	18.6
90	78.1	7.6	14.4
1900	77.3	7.8	15.0
10	73.9	8.8	16.8

The statistics are admitted to be rough, but they suggest that (15), which yields constant relative shares, may not be a very inaccurate model of the actual distribution in the United States. In particular, the share of land seems to have been roughly constant. It would be interesting to see what values would be yielded by the U.S.A. statistics for the various constants and whether they would differ very much from the values yielded by the statistics for the United Kingdom.

Using King's Table XXX, we have the following table showing amounts expressed in billions of dollars of a constant purchasing power (base 1890–1899).

	Total National Income:	Labour's Share:	Land's Share:
1900	17,665.9	13,642.1	1,372.9
1905	20,901.4	15,799.4	1,743.4
1910	24,137.0	17,956.8	2,113.8

The amounts for 1905 are obtained by taking the mid-values between the amounts for 1900 and 1910. The share of land and labour together in 1905 is $17,542.8 billions.

The ratio between the national capital (excluding land) and the national income per annum was about 5, and taking $s=5$ years and using the 1905 absolute shares of the original factors,

$$\beta = \frac{wX}{wX + rL} = \frac{15799.4}{17542.8} = .906,$$

and

$$\gamma = .094;$$

and using the national income for 1910,

$$\alpha = \log\left(\frac{P}{wX + rL}\right) = \log\frac{24,137.0}{17,542.8} = .318,$$

and

$$\eta = .772,$$

and

$$\sigma = 3.2.$$

Tabulating the values of the constants obtained for the United Kingdom and for the United States,

	α	β	γ	σ	η
U.K.	.330	.932	.068	3.34	.84
U.S.A.	.318	.906	.094	3.2	.772

We see that they closely agree as between the two countries, except in the case of γ, the relative share of land in the original factor share. Probably γ is greater in U.S.A. because the relative share of land in U.K. is diminished through competition of land overseas assisting the production of foods and materials imported into the U.K.

For U.S.A. we have

$$P = cL^{.09}X^{.91}s^{.32}, \tag{15''}$$

$$X = CK^{.77}w^{-3.2}, \tag{21''}$$

and

$$-\sigma = -3.2. \tag{22''}$$

It appears that the wage elasticity of the demand for labour in U.S.A. is about as high as in U.K.

Professors P. Douglas and C. W. Cobb (*Theory of Wages*), working by methods different from mine, obtain for U.S.A. a function analogous to (15′′) and also yielding constant relative shares. Their formula is

$$P' = bL^{.75}C^{.25},$$

where P' is "theoretical index of production," L is labour, C is the volume of "fixed capital," and b is a constant. The formula may be criticised on certain theoretical grounds, particularly for leaving the time element (which is represented by my s) and the "circulating capital" out of account. But the equation summarises statistical information very well.

2.8 Some results may be expressed graphically.

Figure 1 is the graph of (15), $P = cL^{\gamma}X^{\beta}s^{\alpha}$, with respect to s, putting

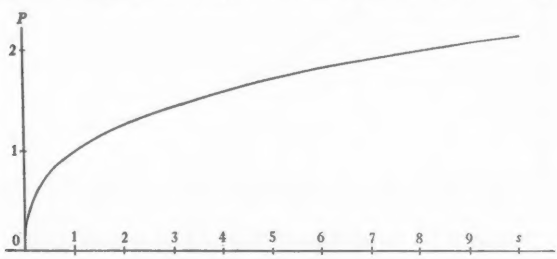

FIGURE 1

$cL^{\gamma}X^{\beta} = 1$ and taking $\alpha = \frac{1}{3}$, which is approximately the value found for the U.K. and the U.S.A.

s	0	.33	.5	1	2	3	4	5	6	7	8	9	10
P	0	.69	.79	1	1.26	1.44	1.59	1.71	1.82	1.91	2.00	2.08	2.15

Figure 2 is the graph of (21), $X = CK^{\eta}w^{-\sigma}$, with respect to w, putting

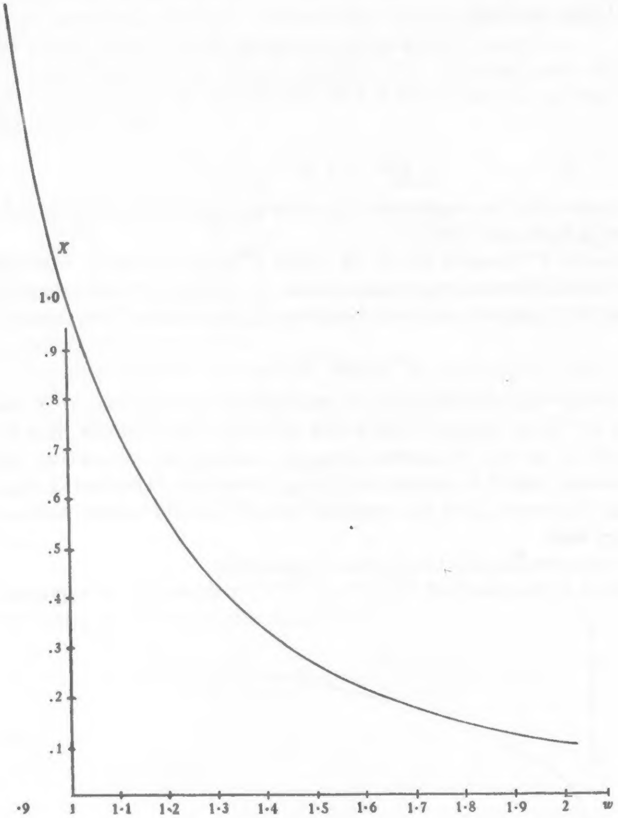

FIGURE 2.—A model Demand Curve for Labor force in the United Kingdom or the United States.

$CK^{.84} = 1$, and taking $\sigma = 10/3$, which is approximately the value found for the U.K. and the U.S.A.

w	0	.9	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
X	∞	1.43	1	.728	.545	.417	.326	.259	.209	.171	.141	.118	.099

Figure 3 is Figure 2 plotted logarithmically and is the graph of $\log X = -10/3 \log w$.

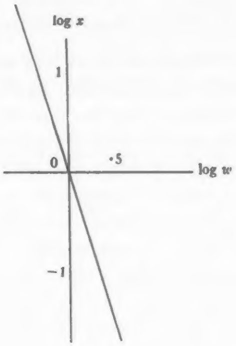

FIGURE 3

2.9 The theory of point input-point output processes is very simple and is not a good model of production, but it enables us to use interesting statistics. It is noteworthy that taking account of the facts of distribution of the national income leads to the same general results as have been reached by more speculative methods, only my econometric analysis yields the results in the more concrete form of empirical functions and numerical solutions.

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A STUDY OF COSTS

By W. A. Tweddle and Richard Stone

The primary object when this work was begun was to find, for several British industries, mathematical functional relationships between volume of employment and volume of production. To this end it was expedient to obtain functional relationships between output-per-head and volume of employment, and output-per-head and volume of production. Fundamentally, however, it was only required to find the function and shape of the labour cost curve of each industry and the movement of this curve through time. The fact that the curve theoretically, could change in shape through time had to be neglected. But this latter was a small point inhibiting the plunge compared with the fear that the theoretical economist would have no tolerance of the results obtained from data so crude. He defines an industry very narrowly and obtains interesting results thereby. But if the statistician is ever to obtain results at all he must be content with what data there are and lump together such things as ships and electrical apparatus and call his industry Engineering. Small wonder that his results are not so neat as those of the theorist.

But once these preliminary difficulties have been put aside, the secondary possibilities of the subject are encouraging. For truly it can be said that no very successful method has yet been evolved for calculating the supply curve of a firm or industry.

In this paper are presented:—

(1) A technique for obtaining supply curves,

(2) Functions giving, for each of the eight industries dealt with, the relation between output-per-head and

  (a) the volume of employment and

  (b) the volume of production,

(3) A function for 60 per cent of the industrial production of Great Britain, together with the trend of rationalisation,

(4) The trends of rationalisation of each of the industries and the changes in the efficiency of labour due to seasonal variations and the like,

(5) Average and marginal labour cost curves, together with their elasticities.

The indices of production, for the eight industries here dealt with and for the country as a whole, are derived from the *Board of Trade Journal*.[1] Their publication in full in Table I was finally decided upon in

[1] See *Board of Trade Journal* for February 1929 for published details of the method of construction of these indices.

view of the following considerations. (1) The production figures presented below are corrected for the number of working days in each quarter, and so are not identical with the crude data appearing in the above-mentioned *Journal.* (2) The method of presentation adopted in the official publication, owing to lack of continuity and frequency of revision, makes reference tedious, even to one possessed of the necessary volumes.

TABLE I
INDICES OF PHYSICAL VOLUME OF PRODUCTION (AVERAGE 1924 = 100)

Year		Mines and Quarries	Iron and Steel	Non-Ferrous metals	Engineering and Shipbuilding	Textiles	Chemicals and Allied Trades	Leather Boots and Shoes	Food, Drink and Tobacco	Great Britain and N. Ireland
1928	I	94.9	104.2	120.0	115.6	105.5	109.0	118.4	97.4	109.3
	II	88.0	103.9	129.4	119.6	102.4	112.6	108.9	102.2	103.6
	III	83.2	97.4	109.9	107.9	91.9	106.3	90.9	105.1	100.2
	IV	91.9	105.2	119.8	111.2	101.2	115.1	91.4	104.8	108.4
1929	I	100.9	114.1	117.4	121.8	103.5	109.9	101.8	98.9	110.6
	II	94.0	119.5	130.6	126.5	101.3	116.3	102.2	110.6	112.0
	III	93.1	112.2	113.1	117.7	90.2	118.7	93.9	110.3	110.7
	IV	100.8	112.2	123.0	120.0	100.7	119.8	97.9	105.7	114.0
1930	I	102.0	113.8	105.7	125.5	92.4	104.9	108.5	96.9	111.0
	II	90.0	98.3	127.4	126.2	80.5	104.0	106.9	108.0	103.1
	III	83.2	78.9	126.9	110.7	70.5	102.8	97.0	106.8	99.5
	IV	91.4	65.8	118.3	106.5	75.2	94.5	95.0	109.4	99.0
1931	I	85.3	68.0	95.5	102.5	72.6	96.1	101.3	99.0	94.6
	II	82.1	64.8	114.2	101.6	77.2	97.2	102.0	104.9	92.1
	III	74.7	60.2	95.4	88.0	69.8	89.5	94.1	107.0	89.3
	IV	85.0	71.1	96.7	88.5	89.0	98.7	100.6	104.5	97.3
1932	I	84.2	71.4	99.2	92.0	92.4	102.8	101.1	95.5	95.0
	II	77.3	66.6	93.4	90.4	86.9	102.8	96.6	101.7	94.3
	III	67.9	60.7	92.9	83.8	75.1	91.3	92.0	96.6	87.4
	IV	80.2	65.7	99.1	87.6	85.6	95.0	95.4	95.8	95.0
1933	I	82.4	71.8	81.3	92.3	87.0	96.0	102.6	88.6	94.8
	II	73.1	80.2	106.8	99.1	88.8	103.5	116.7	104.5	96.7
	III	69.7	83.6	99.6	92.8	88.2	98.9	104.8	101.8	96.8
	IV	84.0	95.1	122.5	101.9	97.3	107.6	103.2	104.2	104.0

The indices of employment are obtained from data published in the Ministry of Labour *Gazette.* The method by which these are produced, of which there are several arithmetical variants, is as follows: first, the annual estimates of the numbers insured in each separate industry in July are interpolated to give estimates for the other months, second, we subtract from these figures the number of persons unemployed during the month. Indices so constructed for groups of industries comparable to the main divisions of the Board of Trade index of production are given in Table III. The inclusion of these data is justified by the fact that, while the crude figures above referred to are of official origin, their compilation into indices of employment is the work of private research. The aggregates were built up by the authors in accordance with details given in Table II, from figures for separate industries produced by Miss O. E. Poulton and Mr. R. W. B. Clarke.

Dividing the production indices by the employment indices, quarterly indices of output-per-head are obtained. These are given in Table IV. This table is interesting in itself, but it is immediately apparent that each coefficient of output-per-head is a function of two things, the

TABLE II

TABLE SHOWING GROUPING OF MINISTRY OF LABOUR GAZETTE INDUSTRIES TO CORRESPOND WITH THOSE OF THE BOARD OF TRADE JOURNAL

Board of Trade	Ministry of Labour
Mines and Quarries	Coal mining; Iron ore and Iron stone mining; Lead, Tin and Copper mining; Stone Quarrying and Mining; Mining and Quarrying not separately specified; Clay, Sand, Gravel and Chalk Pits.
Iron and Steel	Pig Iron (Blast Furnaces); Steel melting and Iron Puddling, Iron and Steel Rolling and Forging; Tin Plates; Iron and Steel Tubes; Wire, Wire Netting, Wire Ropes.
Non-Ferrous metals	Brass, Copper, Zinc, Tin, Lead etc.; Brass and Allied metal wares.
Engineering and Shipbuilding	General Engineering; Engineers' Iron and Steel Founding; Electrical Engineering; Marine Engineering etc.; Constructional Engineering; Motor Vehicles, Cycles and Air craft; Shipbuilding; Electrical wiring and Contracting; Electrical cables, wire and lamps.
Textiles	Cotton; Woollen and Worsted; Silk manufacture and Artificial Silk Weaving; Artificial Silk Yarn; Linen; Jute; Hemp, Rope, Cord, Twine etc.; Hosiery; Lace; Carpets; Textiles not separately specified; Textile Bleaching, Printing and Dyeing.
Chemicals and Allied Trades	Chemicals; Explosives; Paint Varnish, Red and White Leads; Oil, Glue, Soap, Ink, Matches etc.
Leather Boots and Shoes	Tanning, Currying and Dressing; Leather Goods; Boots, Shoes, Slippers, Clogs.
Food, Drink and Tobacco	Bread, Biscuits, Cakes etc.; Grain milling; Cocoa, Chocolate and Sugar confectionary; Food Industries not separately specified; Drink Industries; Tobacco, Cigars, Cigarettes and Snuff.

volume of employment at the moment, and the date at which it occurred, in other words, of the shape of the static curve relating employment and output-per-head, and the position of the curve in time. The reciprocals of output-per-head, or labour costs, will be found plotted against the volume of production for each industry in Figure I for each of the months in question and also for the average of 1924.

TABLE III

INDICES OF VOLUME OF EMPLOYMENT (AVERAGE 1924 = 100)

Year	Mines and Quarries	Iron and Steel	Non-Ferrous Metals	Engineering and Shipbuilding	Textiles	Chemicals and Allied Trades	Leather Boots and Shoes	Food, Drink and Tobacco	Great Britain and N. Ireland
1928 I	81.7	92.2	102.3	105.0	104.9	108.8	100.4	103.0	98.0
II	77.3	92.6	102.6	104.6	103.3	110.1	97.0	104.0	96.3
III	73.6	91.5	102.3	103.2	98.5	110.7	90.5	104.4	93.6
IV	74.2	91.5	102.9	104.1	100.6	111.4	92.3	103.8	94.5
1929 I	80.1	93.6	103.7	106.1	101.5	111.6	92.2	102.5	96.8
II	78.5	95.1	106.6	108.5	101.2	111.7	94.3	104.5	97.3
III	79.7	94.3	107.4	108.1	100.3	112.0	95.1	105.8	97.4
IV	81.1	92.5	109.2	108.4	101.1	111.7	96.2	105.4	98.0
1930 I	82.0	91.1	106.1	106.7	92.5	109.6	94.2	103.0	95.2
II	75.5	84.5	101.1	104.3	83.4	107.4	90.8	102.7	89.9
III	71.3	77.6	96.5	99.7	77.1	105.8	90.1	104.0	85.8
IV	75.9	65.0	91.6	94.6	77.1	102.6	88.3	102.6	84.7
1931 I	71.1	63.0	84.6	89.7	75.6	100.5	86.6	99.8	81.2
II	66.3	61.3	83.0	87.1	77.6	99.5	87.3	101.1	79.8
III	63.5	60.4	82.0	84.3	72.0	98.3	86.5	101.5	76.9
IV	66.9	62.7	83.0	84.4	85.1	97.2	87.2	101.7	81.2
1932 I	67.0	62.3	85.3	83.5	87.9	98.7	88.5	100.7	81.7
II	61.3	58.7	84.3	82.9	82.3	99.5	85.3	102.2	78.5
III	55.4	58.4	83.7	81.6	77.9	100.7	87.3	103.3	75.7
IV	61.5	60.5	86.4	82.0	85.9	100.9	91.3	104.4	79.9
1933 I	63.7	62.3	85.4	80.9	84.2	101.4	90.3	102.9	79.7
II	58.5	65.0	88.6	84.2	84.3	102.6	94.8	105.5	79.8
III	58.1	70.7	91.9	86.8	87.8	104.3	95.3	109.0	82.0
IV	63.5	74.2	94.7	90.8	91.5	106.4	97.1	110.5	85.8

TABLE IV

INDICES OF OUTPUT-PER-HEAD (AVERAGE 1924 = 100)

Year	Mines and Quarries	Iron and Steel	Non-Ferrous Metals	Engineering and Shipbuilding	Textiles	Chemicals and Allied Trades	Leather Boots and Shoes	Food, Drink and Tobacco	Great Britain and N. Ireland
1928 I	116.2	113.0	117.3	110.1	100.6	100.2	117.9	94.5	111.5
II	113.8	112.2	126.1	114.3	99.1	102.3	112.3	98.3	107.6
III	113.0	106.4	107.4	104.6	93.3	96.0	100.4	100.7	107.1
IV	123.9	115.0	116.4	106.8	100.6	103.3	99.0	101.0	114.7
1929 I	126.0	121.9	113.2	114.8	101.9	98.5	110.4	96.6	114.3
II	119.7	125.6	122.5	116.6	100.1	104.1	108.4	105.8	115.1
III	116.8	118.9	105.3	108.9	89.9	106.7	98.7	104.2	113.7
IV	124.3	121.3	112.6	110.7	99.6	107.3	101.8	100.3	116.3
1930 I	124.4	124.9	99.6	117.6	99.9	95.7	115.2	94.1	116.6
II	119.2	116.3	126.0	121.0	96.5	96.8	117.7	105.1	114.7
III	116.7	101.6	131.0	111.0	91.5	97.2	107.7	102.7	116.0
IV	120.4	101.2	129.1	112.6	97.6	92.1	107.6	106.6	116.9
1931 I	120.0	107.9	112.8	114.3	93.7	95.6	117.0	99.2	116.5
II	123.8	105.7	137.6	116.6	99.6	97.7	116.8	103.8	115.4
III	117.6	99.6	116.3	104.4	96.9	91.0	108.8	105.4	116.1
IV	127.1	113.3	116.5	104.9	104.6	101.5	115.4	102.8	119.8
1932 I	125.7	114.6	116.3	110.2	105.1	104.2	114.2	94.8	116.3
II	126.1	112.4	110.8	109.0	105.6	103.3	113.2	99.5	120.1
III	122.6	103.9	111.0	102.7	96.4	90.7	105.4	93.5	115.5
IV	130.4	108.5	114.7	106.8	99.7	94.2	104.5	91.8	118.9
1933 I	129.4	115.2	95.2	114.1	103.4	94.7	113.6	86.1	118.9
II	125.0	123.3	120.5	117.7	105.3	100.9	123.1	99.1	121.2
III	120.0	118.2	108.4	106.9	100.4	94.8	110.0	93.5	118.0
IV	132.3	128.2	129.4	112.2	106.4	101.1	106.3	94.4	121.2

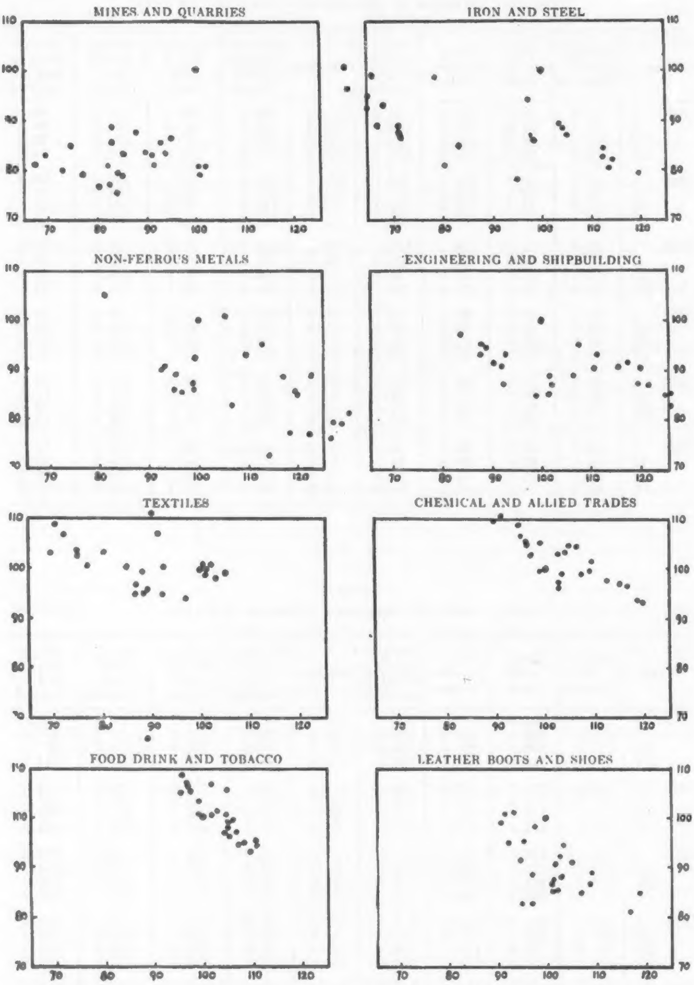

FIGURE I.—Persons employed per unit of output (Labour-costs) plotted vertically against the volume of output plotted horizontally. Crude data for 1924 and the 24 quarters 1928–1933. The means for 1924 are (100, 100) in all cases.

The difficulty is naturally with the factor time. This term is used to mean all those conditions, except the volume of employment, which cause the volume of production to be what it in fact is. The same volume of employment, the technique remaining the same, will not have the same efficiency all the year round. Out-door quarrying and constructional work will be effected by the temperature and humidity of the atmosphere, while the vitality of workers will, in general, fluctuate with the seasons. An equation relating output-per-head with volume of employment and the complicated factor which we call time, (t), may, therefore, be expected to be of the form $Y=\alpha+Ax^m+kt^n$, and there is every reason to believe that the exponents of x and t will be greater or less than one. The problem then becomes that of calculating the values of the constants of this function in order to obtain the most correct value for kt^n, such as to be able to eliminate from the coefficients of output-per-head those determinants which have collectively been called "time." If this can be done correctly, such that $n\sigma^2$ will vanish, then there is good reason to believe that the static curve obtained by eliminating kt^n will be the correct static curve and the problem of calculating supply curves is a step nearer solution. But such hopes must immediately be dashed by explaining that in what follows, t is assumed to be linear. This not only greatly simplifies the work and vastly reduces the hard labour, but is justified by the crudity of the original data.[2]

A function of the form $Y=\alpha+Ax^m+kt$ is, therefore, used here, and the problem becomes one of minimising $\Sigma(y-Y)^2$, i.e., of minimising $\Sigma(y-\alpha-Ax^m-kt)^2$. It is necessary to find the values of the constants. If $P=\Sigma(y-\alpha-Ax^m-kt)^2$, it is apparent that the partial differentials of P with respect to α, A, m, k, will vanish and we have the equations[3]

$$\Sigma(y-\alpha-Ax^m-kt)x^m=0,$$

$$\Sigma(y-\alpha-Ax^m-kt)t=0.$$

If we put $X=x^m$ and get rid of α by changing the origin such that $y'=y-\bar{y}$; $X'=X-\overline{X}$; $t'=t-\bar{t}$ (where $\bar{y}$, $\overline{X}$, $\bar{t}$ are the means of y, X, t, respectively), we get the equations

$$\Sigma(y'X'-AX'^2-kt'X')=0,$$

$$\Sigma(y't'-AX't'-kt'^2)=0.$$

[2] Due recognition for similar work must be given to Mr. R. W. B. Clarke. See *Journal of the Royal Statistical Society*, 1933, Part IV.

[3] With respect to A and k. The equations with respect to α and m are too simple and too complicated, respectively, to serve the purpose of solving to obtain the constants.

TABLE V

TABLE OF FUNCTIONS RELATING OUTPUT PER HEAD WITH VOLUME OF EMPLOYMENT (E) AND VOLUME OF PRODUCTION (P)

Industry	Functions	$n\sigma^2$	Standard Errors		θ	c
			A	k		
Mines and Quarries	(1)	302.6	1283.0	3.358	−20.698	
	(2)	174.7	0.8986	0.121		−17.35
Iron and Steel	(3)	396.0	0.3048×10^{10}	0.2402	−26.074	
	(4)	184.1	0.0049	0.1191		−7.54
Non-Ferrous metals	(5)	2088.1	$0.743+10^{-37}$	0.3811	−22.808	
	(6)	1021.9	28.04	0.2601		−0.471
Engineering and Shipbuilding	(7)	445.7	23.61×10^{6}	0.2575	−8.908	
	(8)	203.6	14.96	0.143		−2.096
Textiles	(9)	265.6	1.657×10^{6}	0.1235	+3.738	
	(10)	154.5	0.5644	0.0885		+3.1
Chemicals and Allied Trades	(11)	339.6	1.48	1.419	+3.116	
	(12)	118.9	48.14	0.0887		+6.279
Leather Boots and Shoes	(13)	800.6	3.069×10^{-400}	0.194	−10.986	
	(14)	323.9	4.556	0.116		−6.894
Food, Drink and Tobacco	(16)	530.3	2.755×10^{67}	0.148	+4.333	
	(15)	99.1	3.454	0.067		+1.9
Employment	(17)	85.7	7.7374	0.1087		

The values θ and c are those required to correct the function of the form $\alpha+Ax^m$ to go through point (100, 100) at the base date 1924.

Let E = employment, P = production, and Y = output per head. Then the functions in Table V are:

(1) $Y=-2553.424+2554.0804E^{0.01}+0.865419t;$

(2) $Y=50.293+4.23069P^{0.6}+0.870843t;$

(3) $Y=78.96+0.23616\times10^{-9}\,E^{5.65}+1.5912t;$

(4) $Y=69.0+0.061069P^{1.4}+1.0092t;$

(5) $Y=125.597-0.01191\times10^{-35}\,E^{18.7}-0.460678t;$

(6) $Y=-266.522+146.096P^{0.2}+0.75835t;$

(7) $Y=116.986-50.97003\times10^{6}\,E^{-3.4}+0.388944t;$

(8) $Y=120.857+88.755P^{0.2}+0.564719t;$

(9) $Y=101.036-4.77402\times10^{6}\,E^{-3.0}+0.463011t;$

(10) $Y=48.6+3.05P^{0.6}+0.482t;$

(11) $Y=96.884+0.13537\times10^{-242}E^{119.0}+0.01789t;$
(12) $Y=-510.393+381.1445P^{0.1}+0.219029t;$
(13) $Y=105.486+0.0055168\times10^{-397}E^{200.0}+0.33877t;$
(14) $Y=-72.272+28.394P^{0.4}+0.22069t;$
(15) $Y=101.385+0.186047\times10^{68}E^{-33.4}-0.28771t;$
(16) $Y=-113.1+33.47P^{0.4}-0.096t;$
(17) $Y=70.400+10.1563E^{0.3}+0.539466t.$

To calculate the values of A and k, let

$$a = \Sigma X'^2, \qquad b = \Sigma t'^2, \qquad c = \Sigma y'^2,$$
$$d = \Sigma X't', \qquad e = \Sigma t'y', \qquad f = \Sigma X'y',$$

and we have the equations $e=kb+Ad$, $f=kd+Aa$. Therefore, $k=(ae-df)/(ab-d^2)$, $A=(de-bf)/(d^2-ab)$, and $n\sigma^2=(y'-Y')^2=c-ke-Af$, where $Y'=kt'+AX'$. If the function were of the form $Y=\alpha+Ax^m+kt^n$, then the fundamental equations would be

$$\Sigma(y'X' - AX'^2 - kT'X') = 0,$$
$$\Sigma(y'T' - AX'T' - kT'^2) = 0,$$

where $T=t^n$ and $T'=T-\overline{T}$.

By a process of trial and error, $n\sigma^2$ is minimised. The data used covered the 24 quarters from 1928 to 1933; Y=the calculated output-per-head, y the original output-per-head, and x equals the employment indices (E) given in Table III and the production indices (P) given in Table I. The calculated functions, together with the standard errors of A and K, are given in Table V. The general form of the Standard Errors[4] of k and A are:

$$\text{Standard Error of } k = \sqrt{\frac{an\sigma^2}{(ab - d^2)21}},$$

$$\text{Standard Error of } A = \sqrt{\frac{bn\sigma^2}{(ab - d^2)21}}.$$

The curves of fit are presented in Figure II.

The output-per-head static curves[5] are important only with regard to their shape, which, however, is assumed not to change through time. It is the most correct shape calculable by the foregoing technique and data. Since $n\sigma^2$ in no case vanishes, and could hardly be expected to vanish by assuming t to be linear, it is necessary, in order to compare one industry with another, to adjust these output-per-head static curves to go through the co-ordinates (100, 100) at the base date

[4] See R. A. Fisher, *Statistical Methods for Research Workers*, Chap. V.
[5] Calculated from the production functions.

(average 1924). The values of these adjustments (C) are given in Table V. If now we calculate $100y/(\alpha+Ax^m+C)$ i.e., the deviations of the actual coefficients of output-per-head from the calculated coefficients of the static curve, we obtain a curve of the fluctuations

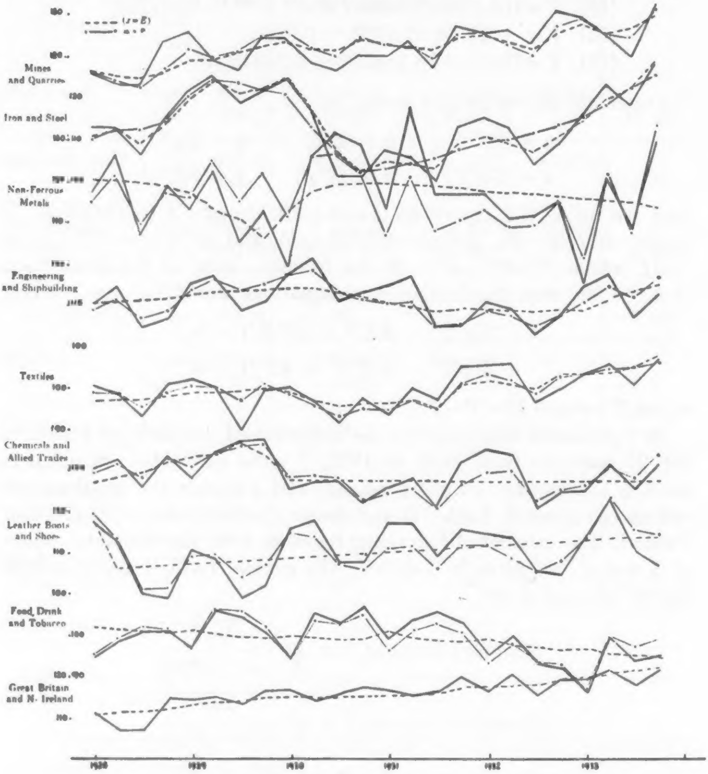

FIGURE II.—Curves of fluctuations in output-per-head during the period 1928–1933. Also the calculated curves plotted from the functions (average of 1924 =100).

through time of the efficiency of labour. These curves are presented in Figure III and the values in Table VI. They are upon the common base of average 1924=100 and are, therefore, comparable. To these efficiency curves are fitted straight lines to obtain the trends of rationalisation during the period.

This chart is of special interest both factual and theoretical. At

the outset, rationalisation appears to be most brisk during the downward phase of depression, and most brisk of all in those industries which are most depressed. The volume of mental exertion of directorates apparently varies inversely with profits, and steel magnates

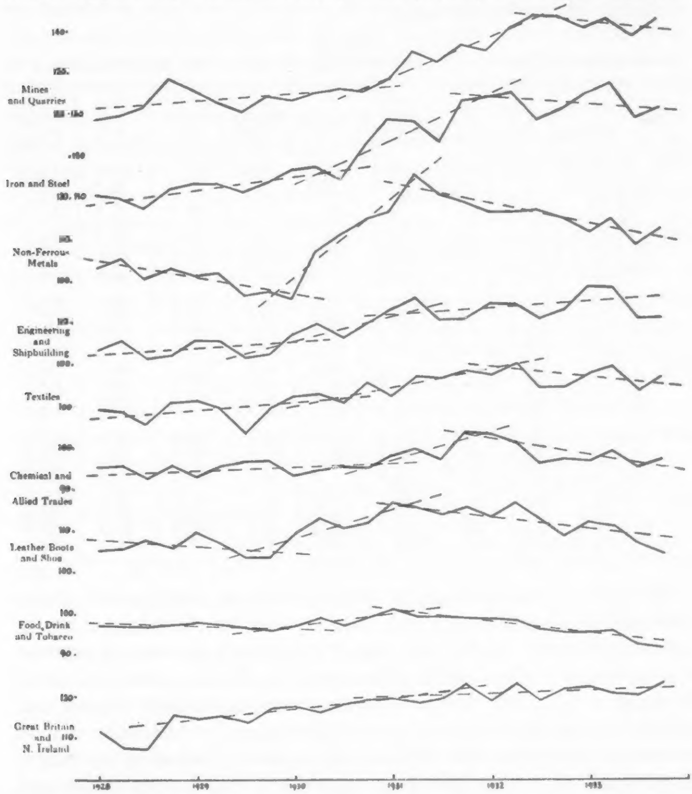

FIGURE III.—Fluctuations in the efficiency of labour due to causes other than changes in the volume of production (average of 1924 = 100).

have apparently seen better days. Part of this increase in efficiency may be due to the movements of a greater proportion of production to the more efficient firms, but this cannot be the full explanation because there would be a movement downwards as the slump was drawing to a close, and this is not in general evidence. Mostly, rationalisation seems to go on at much the same rate as it left off before the slump began, but generalisation is not really possible.

Relative, however, to the volume of production, the amount of fixed capital going into industry appears to be greater, in other words fixed investment falls off at a slower rate than production; and working capital, rather than fixed capital, may again have to take the blame of aggravating disequilibrium.

TABLE VI

INDICES OF FLUCTUATIONS IN LABOUR EFFICIENCY DURING THE PERIOD 1928–33 (AVERAGE 1924 = 100)

Year		Mines and Quarries	Iron and Steel	Non-ferrous Metals	Engineering and Shipbuilding	Textiles	Chemical and Allied Trades	Leather Boots and Shoes	Food, Drink and Tobacco	Great Britain and N. Ireland
1928	I	118.7	110.5	103.3	103.3	99.0	95.2	104.9	96.6	111.8
	II	119.7	109.9	105.7	105.6	98.4	95.4	105.8	96.6	108.1
	III	121.5	107.9	100.4	101.1	95.5	92.5	107.7	96.6	108.0
	IV	128.1	111.9	102.6	102.0	100.2	95.3	105.8	97.1	115.5
1929	I	125.6	113.1	101.3	105.3	100.9	93.2	109.2	97.5	114.7
	II	122.8	113.3	101.9	105.4	99.7	95.4	106.8	97.3	115.5
	III	120.2	111.4	96.5	101.4	92.6	96.4	103.4	96.1	114.1
	IV	123.9	113.7	97.5	102.2	99.3	96.6	103.5	95.8	116.6
1930	I	123.4	116.1	95.7	106.6	102.1	93.2	108.9	96.7	117.3
	II	124.3	117.4	106.7	109.4	102.4	94.3	112.4	98.6	116.2
	III	125.5	114.0	111.6	106.1	100.7	95.4	110.2	97.3	118.1
	IV	124.8	122.1	114.7	109.5	105.5	95.3	111.9	99.0	119.2
1931	I	127.8	128.6	116.8	113.1	102.3	97.9	116.1	100.1	119.4
	II	133.8	128.1	125.3	115.8	107.0	99.2	115.3	99.7	118.5
	III	131.8	123.9	120.5	110.7	106.8	97.3	113.8	99.6	119.8
	IV	135.6	132.7	119.2	110.9	108.1	102.6	115.1	99.1	122.8
1932	I	134.6	134.0	117.0	114.3	107.5	102.3	113.4	98.6	119.1
	II	139.5	134.9	116.6	114.1	109.9	101.4	116.2	98.1	123.6
	III	142.4	128.9	117.3	111.4	104.3	96.2	112.1	96.3	119.4
	IV	142.2	130.9	115.5	113.5	104.2	97.2	108.3	95.2	122.1
1933	I	139.7	134.4	111.9	118.4	107.6	96.9	111.7	95.7	122.1
	II	141.2	137.3	115.0	118.2	109.3	98.6	110.6	95.6	124.5
	III	137.9	129.3	108.7	110.5	104.0	95.6	106.5	92.2	120.8
	IV	141.6	131.6	112.3	111.2	107.2	96.7	104.0	91.3	123.4

Though it is not the purpose of this article, it would be extremely interesting to examine for each industry the two problems, first the substitutability of capital and labour and, second, the relation between the market rate of interest and the marginal efficiency of capital. Since changes in efficiency due to increasing returns of both capital and labour are eliminated by removing the change in efficiency due to volume of production, the efficiency curves here presented give changes in efficiency due to change in the ratio of the two factors capital and labour employed. The curves give, therefore, the rate at which one is being substituted for the other. Thus, if seasonal variations were eliminated from the curves of Figure III, then curves of the substitutability through time of capital and labour would be obtained. If the coefficients of substitutability were functionally related with coefficients of the ratio between the volume of labour and capital, then static substitutability curves, with the elasticity of substitution and the elasticity of demand for the product, would be obtainable. Since the difficulty of changes in the efficiency of labour *qua* labour has been

eliminated, the shape of the efficiency curves is determined by the profitability of introducing more capital. But this latter is itself determined by the relation of the marginal efficiency of capital to the market rate of interest. Since what we wish to examine is the return on capital in relation to its cost, it would be better to use efficiency curves obtained from the employment functions where changes in efficiency due to labour only are eliminated and the increasing physical returns to capital remain. The efficiency curves would then show the fluctuations in output-per-head due, first, to the changes in the ratio of the amount of capital and labour employed and, second, to the increasing returns from capital as an increasing amount was used. Marginal efficiency would diminish more rapidly than if the production functions were used. But this problem cannot be treated here, and one must be content with the straight line trends of the efficiency curves here given. They are particularly interesting, however, in the light of the following data. Interest rates in Britain rose gradually from the first quarter of 1928 to about 5.5% in the last quarter of 1929. Thereafter they fell considerably, until the crisis in the last six months of 1931, when they again rose for a short time. They were again low in the second quarter of 1932, from which they gradually fell still lower to the lowest level for a generation at the close of the period. These data, together with the slopes and changes in the slopes of the straight line efficiency trends, are such as to encourage speculation on the profitability of an industry at any date.

To the static theorist and the theorist of the static short period, Figure III may not be very encouraging. It shows how the static curve moves through time, and how cost diverges from what the static curve would have us believe it to be. Almost in no case is the continuous curve horizontal between one quarter and the next—Food, Drink and Tobacco, being a conspicuous exception. An industry does not appear, in actual practice, to stay upon the same supply curve for any length of time. But at best we have only calculated an average prime cost curve, which is in fact only a labour cost curve and, therefore, the fluctuations in cost due to seasonal vagaries may be more violent than is warranted. Nevertheless, it is just the seasonal changes in the volume of output which the static short period theorist would have us believe occur upon a stable static cost curve, and in the short period the labour cost curve approximates the supply curve very closely. Next, the trend of rationalisation is never horizontal—as the static short period theorist tries to persuade himself—and is most positively oblique during depressions—just at the time when, theoretically, it should be most nearly horizontal.

Thus for no length of time is cost solely a function of output, and for

no length of time does an industry operate upon the same supply curve.

But it must be remarked that the slopes of the trends of efficiency changes are never very great, and, therefore, it is quite possible for the

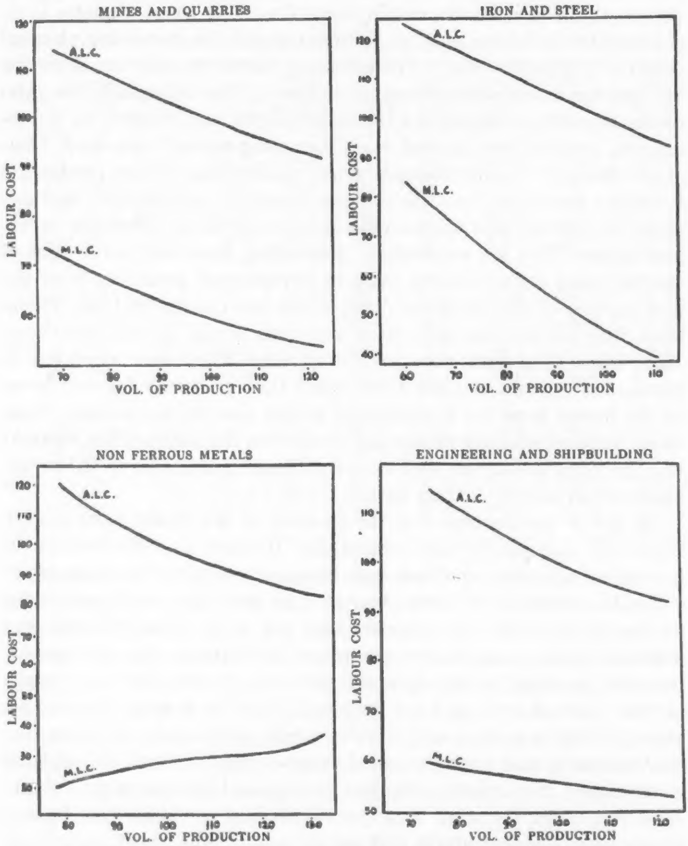

FIGURE IV.—Average and marginal Labour Cost Curves.

static theorist to argue that they are unimportant and that rationalisation in fact takes place by vertical jumps interspaced by periods of horizontal quiescence. This is a matter for his own conscience. However, it must be pointed out that the seasonal variations in efficiency may be partly due to the seasonal variations in the volume of pro-

duction, since kt does not give a perfect elimination of the factor "time."

It is interesting to note, in passing, that the trends of efficiency are very similar in Mines and Quarries and in Iron and Steel and, further, that the decreasing supply price which is in general evidence in Figure IV is an increasingly decreasing supply price in the case of Mines and Quarries, Iron and Steel, and Textiles, and a decreasingly decreasing supply price in the other industries. One would hardly have expected Engineering to be in this latter group.

The static curves of interest are those of labour costs plotted against the volume of production—labour cost curves which might be regarded as average prime cost or short period supply curves. Since in no case does $n\sigma^2$ vanish, the curve of the function $Y=\alpha+Ax^m$ does not —as was mentioned previously—go through the co-ordinates (100, 100). But by adding a quantity C (Table V) such that $\alpha+C=\beta$ the function $(Ax^m+\beta)^{-1}$ of the labour cost curve may be made to do so. The function of the marginal labour cost curve, and the functions of the elasticities of the average and marginal labour cost curves, will be as follows:

$$M = \frac{Ax^m(1-m)+\beta}{(Ax^m+\beta)^2},$$

$$E_A = \frac{Y}{Amx^m},$$

$$E_M = \frac{Ax^m(1-m)+\beta}{Amx^m(1-m)(1-2MY)}.$$

The average and marginal curves are presented, over the relevant range, in Figure IV and their elasticities in Table VII.

Now it is immediately necessary, in defence, to point out to the critic that these are not curves of single individual firms producing homogeneous products, but of vast industries, such as Engineering, producing products almost as diverse as the celebrated Walrus' ships and sealing-wax.

Most of the curves, however, have come out much as one would expect. Increasing returns are evident; the elasticities are almost constant; the minimum of the average curve is almost beyond reach in practice; there has apparently been no deficiency of any of the factors of production during the period; labour, probably the most potent, has been in fact in distressing abundance.

It must be remembered, however, that in the cases presented, competition will be extremely imperfect, the demand curve will be falling, and the interdependence of demand and supply must be considerable. Competition will, in the nature of the case, be experienced only from

the general market and from the foreign particular market. Also, it is not correct to assume that it is possible for an industry to produce at any point within the range 80–100 per cent, at any moment, but that some time may be required for it to readjust itself to the new level. With a given change in output, the problem of whether or not increasing returns are operating, as is suggested by the static curves,

TABLE VII
THE ELASTICITIES OF THE AVERAGE AND MARGINAL LABOUR COST CURVES

When x equals	Mines and Quarries		Iron and Steel		Non-Ferrous Metals		Engineering and Shipbuilding	
	E_A	E_M	E_A	E_M	E_A	E_M	E_A	E_M
80	2.603	3.280	2.271	2.865	1.196	0.771	2.116	1.335
90	2.539	3.235	2.035	2.569	1.286	0.822	2.184	1.378
100	2.487	3.134	1.853	2.342	1.362	0.868	2.243	1.414
110	2.441	3.077	1.711	2.165	1.432	0.910	2.295	1.447
120	2.401	3.027	1.595	2.021	1.492	0.947	2.341	1.476

When x equals	Textiles		Chemicals and Allied Trades		Leather Boots and Shoes		Food, Drink and Tobacco	
	E_A	E_M	E_A	E_M	E_A	E_M	E_A	E_M
80	3.704	4.661	1.467	0.823	1.292	1.101	1.061	0.968
90	3.565	4.488	1.565	0.878	1.348	1.145	1.127	0.983
100	3.448	4.340	1.655	0.928	1.395	1.184	1.184	1.019
110	3.350	4.218	1.734	0.972	1.437	1.217	1.233	1.055
120	3.264	4.109	1.841	1.031	1.473	1.247	1.277	1.089

must be determined from an examination of the static cost curves and the efficiency curves conjointly. It is important to recognise in reference to these cost curves that we have no knowledge of how the labour of these curves is associated with capital, or how the ratio changes as production increases, or how to allocate the increasing returns, apparent in the Figure, to labour and capital respectively, or what part is due to organisation as such or to the supply of the other factors, or how far the increasing returns shown are due to internal or to external economies, or how far this latter effects the labour efficiency curve. Nevertheless, it would be extremely interesting to compare carefully the movements in the production indices with movements along the static cost curves and movements in the labour efficiency curves. It might be possible to obtain thereby some estimate of how long it took an industry to adjust itself efficiently to a change in the volume of output at any particular date. Associated with prices and demand (and it is hoped that it may be possible to produce some demand curves shortly) this would be a very interesting study. Also it might be mentioned, finally, that if seasonal variations were eliminated from the efficiency curves, a comparison of the static cost curves and these labour efficiency curves with the market rate of interest might be enlightening. By obtaining a functional relationship between the market rate of interest and the coefficients of substitutability, a static curve relating

these two might be obtained, together with a long period trend due mainly to general technical progress. This might be a straight line trend of constant slope. If this were correct, it would be possible to estimate

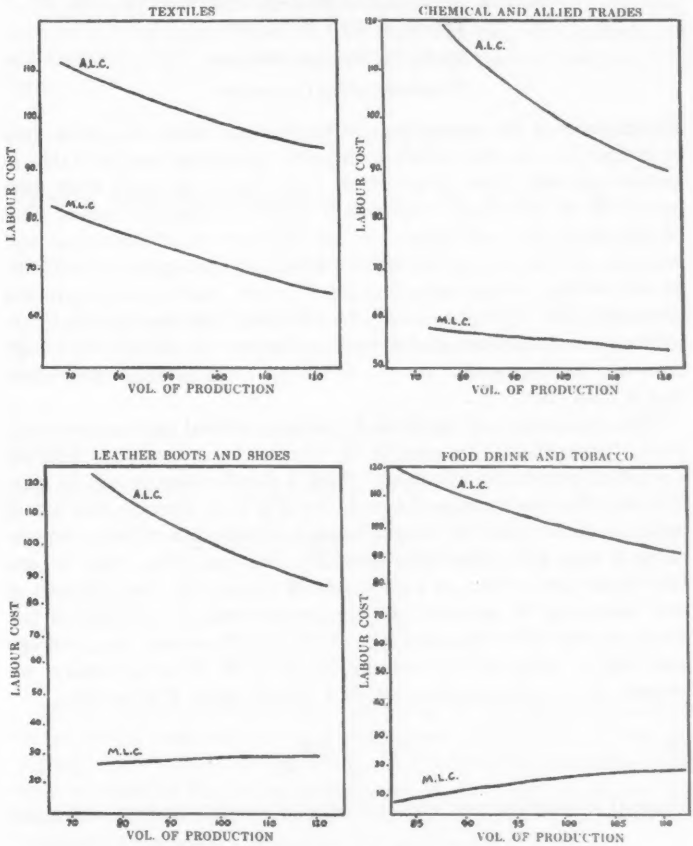

FIGURE V. Average and Marginal Labour Cost.

future costs of production in so far as future movements of the market rate of interest could be estimated.

In conclusion, our best thanks are due to Mr Colin Clark for his continual encouragement and advice.

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COST CATEGORIES AND THE TOTAL COST-FUNCTION

SECOND REPORT OF THE ECONOMETRICA COMMITTEE ON SOURCE MATERIALS FOR QUANTITATIVE PRODUCTION STUDIES[1]

By E. H. Phelps Brown

Chairman of the Committee

Statements of the proportions in which total costs at a given rate of output are divided between different categories are available in greater quantity than those which show how total costs vary with output. It is, therefore, of interest to point out that the former kind of statement does not merely set out the state of affairs at one and only one rate of output, but diffuses a halo of light upon the behavior of costs within a range extending some little way above and below the given rate. The light is seen only by reflection from assumptions that we choose to introduce, and it becomes dimmer the further we depart from the starting-point; yet the following considerations may show that it does exist.

Thus, (a), costs may be divided simply into fixed and proportional, where "proportional" is used in its strict sense as meaning "bearing a constant proportion to output." Such a classification cannot be suitable over the whole range of output, for if it were, average cost would continue to fall until the output became infinite; but within a certain range it may give a fair approximation. Suppose, then, that we are told of one factory that, at a given rate of output, the fixed element of cost makes up 40 per cent and the proportional 60 per cent of the whole: we can infer that at a rate of output 10 per cent less, average cost will be increased in the ratio of 94 to 90. More generally, we assume as an approximation within a certain range the function,

$$C = \frac{a}{Q} + b, \tag{1}$$

where C is average cost and Q is rate of output; and we are given

$$\frac{a}{Q} : b = p_1 : p_2, \tag{1.a}$$

where p_1 and p_2 are the proportions of cost falling into the two categories, and so correspond to 40 per cent and 60 per cent in our example. We may or may not know the actual value of Q, but we are in any case free to put it equal to unity, since the choice of the unit of output

[1] See the Introduction on p. 123 in the April, 1936 issue of Econometrica.

is arbitrary. We may then use the relation (1.a) to determine b in terms of a, that is, to determine our function save for a further arbitrary choice, that of the unit of cost.

(b) Where the classification distinguishes an element of total cost, such as indirect labor, which we may expect to vary with output, but not in direct proportion, we may similarly determine a function of the form

$$C = \frac{a}{Q} + b + cQ, \tag{2}$$

where for some special magnitude of the output Q_1 we are given

$$\frac{a}{Q_1} : b : cQ_1 = p_1 : p_2 : p_3. \tag{2.a}$$

(c) The suitability of this form may be judged by the plausibility of its implication that the output of minimum average cost will be Q_m, where

$$Q_m = \sqrt{\frac{p_1}{p_3}} \cdot Q_1. \tag{2.b}$$

For (2) is a minimum for Q_m where

$$-\frac{a}{Q_m^2} + c = 0, \qquad Q_m = \sqrt{\frac{a}{c}}.$$

But we know

$$\frac{a}{cQ_1^2} = \frac{p_1}{p_3}.$$

(d) In the last clause we have proposed to compare the value obtained for Q_m with our reasonable expectation of its value. When this expectation is sufficiently definite to allow a quantitative estimate, we may use another kind of approximation. If, for instance, we are told that the proportions given apply to an output which is 85 per cent of "capacity," we might assume

$$Q_m = \frac{100}{85} Q_1.$$

We then could put

$$C = \frac{a}{Q} + b + cQ + dQ^2, \tag{3}$$

and determine the constants (in terms of a) from the equations,

$$\text{(i)} \qquad b \div \frac{a}{Q_1} = p_2 \div p_1,$$

$$\text{(ii)} \qquad [cQ_1 + dQ_1^2]\frac{Q_1}{a} = \frac{p_3}{p_1},$$

$$\text{(iii)} \qquad -\frac{a}{Q_m^2} + c + 2dQ_m = 0,$$

where the third equation expresses the condition that average cost shall be a minimum for Q_m.

(e) If we have the tripartite division of total cost into fixed, proportional, and otherwise variable elements, and if we are given the proportions in which cost is divided at each of two levels of output, then at each level the relation of form (2.a) yields two equations, so that we have four equations in all. Unless, therefore, our system is to be overdetermined, we must propose a function having five parameters.

(f) If we take the elasticity of supply at the point studied to be defined by (proportionate change in output)÷(proportionate change in average cost), then, in the example first considered, where we supposed a fixed element to be 40 per cent and a proportional 60 per cent of the whole, we can say at once that the elasticity of supply is $-(1+60/40)$. For we have

$$\eta_s = \frac{dQ}{Q} \div \frac{dC}{C} = -\frac{Q^2}{a}\cdot\frac{\frac{a}{Q}+b}{Q},$$

$$= -\left(1 + \frac{bQ}{a}\right),$$

and from (1) we have

$$\frac{bQ}{a} = \frac{p_2}{p_1}.$$

If the function be of the form (2), we have

$$\eta_s = \frac{Q^2}{-a+cQ^2}\cdot\frac{a+bQ+cQ^2}{Q^2},$$

$$= \frac{a + \frac{p_2}{p_1}a + \frac{p_3}{p_1}a}{-a + \frac{p_3}{p_1}a} = \frac{1}{-p_1+p_3}.$$

We are thus reminded that a function of the form (2) will be increasing or decreasing at the point studied according as p_3 is greater or less than p_1.

Let us now consider some samples of the material available.

(1) The division of all costs into fixed and variable has been applied by Dr. Otto Rostrup, who serves as economic-technical adviser to some fifty Danish enterprises, and who has based upon this experience, and more especially upon the records of the years 1926–28, the figures given in Table XIII.[2] The percentages apply to a level of output which Dr. Rostrup terms "normal employment," "the employment which enterprises within a certain branch can (or ought to) reckon on obtaining as an average for a number of years under quiet market conditions." If the concern is to maintain this level of output, or be always able quickly to revert to it, then a certain framework must be maintained, and all the costs necessary to this framework Dr. Rostrup calls "general costs." The other element, which he calls "particular costs," comprises all costs which will vary as output varies above or below the level of normal employment. Thus, depreciation must in most cases be reckoned a general cost, since many machines suffer less from use than from disuse, but in some cases it is reckoned half as general and half as particular. The repair of machinery falls among the particular costs, that of buildings among the general. Of interest charges, the greater amount must be reckoned as general cost, only that amount being particular which will vary with changes in stock carried and in advances to customers; and similarly with charges for power. Dr. Rostrup thus arrives at the following figures:

(2) In the absence of more definite information, guidance as to the relative parts of fixed and variable costs in different trades may be sought in figures such as are collected in a Census of Production, showing the number of workers in relation to the total "value added" or net product of each trade.

(3) The Committee on Industry and Trade, appointed under the Chairmanship of Sir Arthur Balfour by the British government in 1924, gathered from various firms and trade associations data arranged to show the proportions in which total cost was divided between different categories, before and after the war. The information is made available in the Committee's reports.[3]

[2] Om "Faste" og "Løbende" Udgifter i Industrien, foredrag i Dansk Ingeniørforening d. 6 Nov. 1929; Copenhagen, 1930.

[3] Committee on Industry and Trade. *Further Factors in Industrial and Commercial Efficiency*, Part II of a *Survey of Industries*. H. M. Stationery Office, London, 1928.

TABLE XIII

DIVISION OF COSTS BETWEEN THE CATEGORIES "GENERAL" AND "PARTICULAR" IN CERTAIN DANISH ENTERPRISES, ACCORDING TO THE ESTIMATES OF DR. OTTO ROSTRUP

Type of enterprise	"General" costs as percentage of total costs (including raw materials) at normal employment
	ca.
Electrical generating station	65
Gas works	56
Printing (without paper-making)	50
Button factory	50
Briquet plant	37
Special plants, pottery	32
Mixed printing and paper-making	32
Bicycle factory	28
Electrical fittings factory	28
Special factory in metal industry	27
Glassworks	27
Bolt and nut factory	26
Tool factory	26
Iron foundries with mixed production	26
Iron foundries with mass production	26
Manufacture of files	24
Manufacture of agricultural machinery	23
Machine factories with mixed production	22
Bookbinding	18
Rope works	18
Manufacture of cattle foods	17

TABLE XIV

DIVISION OF COST OF REFINING SPIRITS IN EIGHT POLISH REFINERIES, 1928

Category of cost	Average Cost per Hl., in zloty		
	Private factories	State factories	General average
Fuel	1.60	2.46	2.01
Other subsidiary materials	1.02	0.42	0.74
Administrative labor (including social charges)	0.37	0.03	0.21
Skilled labor (including social charges)	2.58	1.93	2.27
Unskilled labor (including social charges)	1.57	2.31	1.91
Repairs, replacements, and insurance	0.63	0.67	0.65
Capital charges	0.34	2.36	1.29
Taxes, other than charges on labor	0.74	0.33	0.54
Other costs	1.78	0.29	1.08
Total of gross costs	10.63	10.80	10.70
Receipts from by-products	2.71	0.41	1.63
Total of net costs	7.92	10.39	9.97
Depreciation	3.45	0.29	1.97
Total cost	11.37	10.68	11.04

TABLE XV

DIVISION OF COST OF PRODUCTION OF BAR IRON IN THREE POLISH PLANTS, 1927–29
(From Investigations by the Warsaw Institute for Economic Research)

Category of Cost	Blast Furnace					Martin Oven					Rolling Mill				
	1927		1928		1929	1927		1928		1929	1927		1928		1929
	1st half	2nd half	1st half	2nd half	1st 3 mths.	1st half	2nd half	1st half	2nd half	1st 3 mths.	1st half	2nd half	1st half	2nd half	1st 3 mths.
1. Materials	142.96	146.86	143.70	146.44	151.94	175.47	191.04	182.57	180.99	192.01	244.24	271.17	269.52	271.84	284.20
2. Fuel	—	—	—	—	—	6.26	6.26	7.10	7.34	8.00	2.71	2.91	3.12	3.32	4.05
3. Labor, including taxes	8.91	9.81	10.40	11.17	12.01	6.11	6.41	7.59	8.27	8.34	11.74	12.33	12.93	13.67	15.54
4. Clerical labor, incl. taxes	1.46	1.31	1.50	1.64	1.84	0.84	0.83	0.95	0.95	0.98	2.10	1.76	2.24	2.07	2.22
5. Other working costs	14.94	17.98	15.59	16.28	17.21	26.18	28.09	29.78	32.39	29.99	17.34	20.42	20.24	19.24	19.81
6. Overhead costs	3.01	6.28	5.52	5.59	5.28	4.34	6.57	6.62	6.73	7.74	7.16	5.72	8.01	8.15	10.44
7. Total gross costs	171.28	182.24	176.71	181.12	188.28	219.20	239.20	234.61	236.67	247.06	285.29	314.31	316.06	318.29	336.26
8. Subsidiary receipts	2.34	2.46	3.08	3.50	3.84	4.02	3.98	3.92	3.52	3.85	9.75	9.95	9.86	9.72	10.12
9. Total net costs	168.94	179.78	173.63	177.62	184.44	215.18	235.22	230.69	233.15	243.21	275.54	304.36	306.20	308.57	326.14

All costs are in zloty. Those in the Blast Furnace column are given per tonne of pig-iron made; those in the Martin Oven column per tonne of the Martin block; those in the Rolling Mill column, per tonne of bar iron. In the Blast Furnace, materials comprise ore and coke; in the Martin Oven, pig-iron and other iron (fragments); in the Rolling Mill, Martin block and blooms.

(4) From the plentiful material provided in the studies of the Warsaw Institute for Economic Research we cite the figures presented in Tables XIV and XV.[4] The data of Table XV were obtained by direct investigation into the accounts of the three stages of production in each of three plants.

(5) The division of costs into fixed and proportional is illustrated in another way in Table XVI, in which are shown data collected by M. L. Kahn, Ingénieur du Génie Maritime, and exhibited in June 1930 to a conference of the French Association Technique Maritime et Aéronautique.[5] To give a unified account of these figures, M. Kahn suggests the hypothesis that the total costs of air transport are composed of two elements only, one directly proportional to the traffic carried, the other directly proportional to the length of the line. We then have the following formulas:

$$\text{Unit cost} = \frac{\text{Total annual expenses}}{\text{Total km.-tonnage carried during year}},$$

$$p = \frac{D}{T}.$$

$$\text{Utilization} = \frac{\text{Total km.-tonnage carried during year}}{\text{Length of line}},$$

$$u = \frac{T}{L}.$$

Then,

$$p = \frac{k_1}{u} + k_2,$$

or

$$D = L \cdot k_1 + T \cdot k_2.$$

Fitting hyperbolas to the data for 1927 and 1928, M. Kahn obtains the following results:

1927 $k_1 = 3{,}500$ francs per km. of line,

$k_2 = 24$ francs per tonne-km.

1928 $k_1 = 3{,}700$ francs per km. of line,

$k_2 = 24.50$ francs per tonne-km.

[4] Instytut Badania Konjunktur Gospodarczych i Cen. Sprawozdania i Przyczynki Naukowe. No. 6, Koszty Oczyszczania Surówi Spirytusowej, 1929, Tablica XIII, p. 8. No. 3, Koszty Produkcji Zelaza Sztabowego, 1929, Tablica VII, p. 4; Tablica X, p. 7; Tablica XIX, p. 10. I am indebted to Mr. A. Langnas for invaluable help given in the translation of Polish materials.

[5] Extrait du compte rendu de la session plénière, Juin, 1930, Paris, Imprimerie Chaix, 1930.

M. Kahn also records the application of a similar hypothesis to the French railways. Using the data published by the Ministère des Travaux Publics for six lines (Est, Midi, P. O., P.L.M., État, Nord) for the years 1911, 1912, 1913, he finds

$$k_1 = 23{,}500 \text{ francs}, \qquad k_2 = 0.0365 \text{ francs}.$$

For the same railways with the addition of the Alsace Line, he finds for 1927

$$k_1 = 110{,}000 \text{ francs}, \qquad k_2 = 0.16 \text{ francs}.$$

TABLE XVI

INDEX OF UTILIZATION AND REALIZED COST PER TONNE-KM., IN THE FRENCH AIR-LINES, 1925–28

	Companies						
	A	B	C	D	E	F	G
1925							
(i)	1,025	192	83	50	39	109	561
(ii)	27	42	68	110	149	53	
1926							
(i)	203	85	61	62		85.5	718
(ii)	48	58	98.5	95.5		74.5	
1927							
(i)	201	150	33	91		75	653
(ii)	49	40	120	68		70.5	
1928							
(i)	279	230	29	127		191	620
(ii)	39	36	179	54		43.5	

In each year, (i) is the index of utilisation: tonnes per kilométre carried per annum;
(ii) is the average cost experienced per tonnekilométre, in current francs.

A: Air-Union. Reduced in 1925 to Paris-London, it later took over Antibes-Ajaccio, and now (1930) also works Paris-Marseilles.

B: Société générale de transports aériens: Paris-Brussels-Amsterdam-Berlin.

C: Compagnie générale d'entreprises aéronautiques; now (1930) Compagnie générale aéropostale. It has progressively extended its line from Morocco to South America.

D: Compagnie internationale de navigation aérienne.

E: Compagnie aéronavale. Antibes-Ajaccio; in 1926 absorbed by the Air-Union.

F: French air lines as a whole. In making the average for 1928, the figures of the Aéropostale have been omitted, as a part of its line, the crossing of the South Atlantic, is worked by ship. But if these figures be included, the index of utilisation will appear as 86.25 and the cost as 73.50 frs.

G: Index of wholesale prices, 1914 = 100, average for the year.

(6) We have a second French study of the costs of air transport, a study remarkable, and perhaps indeed unique, in its building up of a supply function out of a detailed knowledge of the component elements of cost, namely "Détermination et calcul du Prix de Revient des Transports Aériens," by M. Louis Bréguet, Ingénieur-Constructeur.[6] M. Bréguet does not use recorded statistics, but draws upon his experience

[6] Librairie Aéronautique. Paris: Étienne Chiron.

as a builder of aeroplanes to express and estimate the different factors in the realms of engineering, organization, and prices, that make up the long-period supply function. To follow the making of this estimate requires some effort of attention, but the effort is rewarded by the understanding of a function which, though it will not answer every question which the economist would like to put, contains in itself an unusually detailed statement of conditions of supply. We shall first, (A), follow M. Bréguet in his construction of a function of total cost, and here help may be given by reference to the table in which the different symbols used are brought together. We shall then, (B), segregate those variables by changes in which the amount of service offered may be varied, and for the others we shall substitute the numerical values provided by M. Bréguet's estimate or suggestion: we shall then be able to study the course of marginal and average cost. Finally, (C), by relating the proportionate change of a coefficient to the resultant proportionate change in average cost, we shall examine the parts severally played in cost-determination by the several coefficients.

TABLE OF SYMBOLS USED BY M. BRÉGUET

Symbol	Value given	Dimension	Definition
F_u	Derived from other terms	kg.	Fret utile or pay-load of one plane
F		kg.	Gross carrying capacity of one plane
a	0.85	ratio	Proportion of F not absorbed by coach-work and fittings.
P	5000	kg.	Total weight of one fully loaded plane
P_p	Derived	kg.	Weight of body of one plane without engines or equipment.
P_m		kg.	Weight of engines of one plane
P_c		kg.	Weight of load of fuel and oil for one plane
P_e		kg.	Weight of equipment and crew of one plane
α	0.35	ratio	Coefficient of lightness of construction, $=P_p \div P$
W_0	Derived	nominal h.p.	Aggregate power of engines of one plane
q	1.50	kg. per nominal h.p.	Coefficient of weight of type of engine used
W_e	Derived	effective h.p.	Power needed by one plane for level flight at air-speed V
V	160	km. per hr.	Average cruising air-speed
$tg\cdot\phi'$	0.18	ratio	Index (inverse) of aerodynamic performance of type of plane used
x	1.50	ratio	Ratio of minimum safe equipment of power, to power sufficient for level flight at speed V
m	0.280	kg. per effective h.p. per hr.	Rate of consumption of fuel and oil at air-speed V

TABLE OF SYMBOLS USED BY M. BRÉGUET (*Continued*)

Symbol	Value given	Dimension	Definition
v	zero	kms. per hr.	Average speed of prevailing contrary wind
E	400	km.	Length of stage flown
N		no.	Total number of stages flown by the service during the year
n'		no.	No. of planes flying on any one day
n''		no.	No. of planes in reserve on any one day
n'''		no.	No. of planes undergoing overhaul on any one day
J	120	days	Time occupied by overhaul of one plane
H	300	hours	Time flown by plane between two successive overhauls
λ	1	ratio	Coefficient reducing air-speed to ground-speed
$1/A$	1/5	ratio	Proportion of fleet withdrawn as obsolete in each year
$1/k$	1/4	ratio	Proportion of original cost of plane represented by cost of one overhaul
x_p	12.50	francs-or per kg.	Purchase-price of plane
x_m	50.00	francs-or per nominal h.p.	Purchase-price of engine-power
x_c	0.75	francs-or per kg.	Purchase-price of fuel and oil
M	600	hours	Working life of an engine
$1/k'$	1/6	ratio	Proportion of original cost of engine represented by cost of one overhaul
H'	100	hours	Time worked by an engine between two successive overhauls
G	200,000 250,000	francs-or	Fixed element of general expenses
G'	0.20	francs-or per tonne-km.	Coefficient for variable element of general expenses

(A) M. Bréguet first constructs a formula to show the equipment needed to supply a given service, and then considers the costs occasioned by working this equipment.

(1) *The pay-load supplied by a given equipment.*—For a single plane we have the formula

$$F_u = aF = a[P - P_p - P_m - P_c - P_e],$$

where F_u is the pay-load (fret utile) in kg.; a is the proportion of gross available tonnage not absorbed by fittings and coachwork; P is the total weight in kg. of the fully equipped and loaded plane; P_p is the weight in kg. of the body, without engines or equipment; P_m is the weight in kg. of the engines of one plane; P_c is the weight in kg. of the initial load of fuel and oil; and P_e is the weight in kg. of the equipment

and crew. We must now express the different elements of F_u as functions of P.

(i) P_p. This may be expressed as αP, where α is the "coefficient of lightness of construction." In general, for similar types of machines, α increases with the size and weight of the plane. For planes of medium size, technical progress since the War has reduced α from the region of 0.4 to that of 0.3.

(ii) P_m. We may write

$$P_m = W_0 \cdot q,$$

where W_0 is the aggregate nominal horsepower of the engines, and q is their weight in kg. per nominal h.p. The effective power needed, W_e, for level flight at a speed of V km. per hour (which we shall take as a constant), is given by

$$W_e = \frac{P \cdot V \cdot tg\psi'}{270},$$

where $tg \cdot \psi'$ is the index of aerodynamic performance: the smaller $tg \cdot \psi'$, the greater the tonne-kilométrage achieved per unit of power generated. We have now to consider what power must be supplied to cover the reserve needed for climbing and for the margin of safety, and we may express this by putting

$$W_0 = \chi W_e,$$

where χ is a coefficient greater than unity. We then have

$$\begin{aligned} P_m &= W_0 \cdot q, \\ &= \chi \cdot W_e \cdot q, \\ &= \chi \frac{P \cdot V \cdot tg\psi'}{270} \cdot q. \end{aligned}$$

(iii) P_c. The relation between P_c and P is complicated: the rate of consumption needed at any moment to maintain the speed V varies with the momentary total weight of the plane, and even, therefore, if we assume the consumption of fuel and oil to be simply proportional to the h.p. per hour to be developed, the load of fuel and oil that must be present at the start will not be simply proportional to the distance to be flown. We have also to consider the effects of the prevailing winds. After developing a precise but complicated formula, however, M. Bréguet suggests that for flights not exceeding 1500 km., P_c may be given approximately by the formula

$$P_c = P \cdot \frac{m \cdot tg \cdot \psi'}{300} \cdot \frac{V}{V - v} \cdot E,$$

where m, taken as a constant, is the consumption of oil and fuel in kg. per h.p. per hour,

v, in km. per hour, is the average speed of a prevailing contrary wind,

E, in kms., is the length of the stage to be covered.

(iv) P_e. For commercial planes of middle size and flying stages of not more than 1500 km., M. Bréguet considers that P_e may be given by the equation

$$P_e = 100 + 0.03P.$$

Bringing these elements together, we may now express the pay-load as a function of the total weight of the plane in the following equation:

$$F_u = aP\left[1 - \alpha - \chi \cdot \frac{V}{270} \cdot tg \cdot \psi' \cdot q - \frac{m \cdot tg\psi'}{300} \cdot \frac{V}{V-v} \cdot E - \frac{100}{P} - 0.03\right].$$

Total pay-load supplied per annum by a given service is, in tonne-kms.,

$$\frac{F_u \cdot N \cdot E}{1000},$$

where N is the total number of stages flown per annum and division by 1000 is needed to reduce kgs. to tonnes. Substituting for all terms save N the values given in the table, we have,

Total pay-load supplied $= 497.76\ N$ tonne-kms. per annum.

(2) *Total cost of offering a given service.*—If day by day there are n' planes in use for flying, there must also on any one day be n'' planes held in reserve, and n''' withdrawn for overhaul. M. Bréguet considers that n'' has the minimum value 2, but stands in no general relation to n'; n''', however, may be related to n' as follows:

$$\frac{n'''}{n'} = \frac{\text{no. of days needed to overhaul one plane } (= J)}{\text{no. of days' flying done by one plane between two successive overhauls}}.$$

The denominator is given by

$$365 \times \frac{\text{no. of hours' flying done by one plane between two successive overhauls}}{\text{total no. of hours of flight per plane per year}}.$$

This may be written

$$365 \times \frac{H}{\dfrac{N}{n'} \cdot \dfrac{E}{\lambda V}},$$

where H is the no. of hours of flight achieved by one plane between successive overhauls;

λ is the coefficient reducing airspeed to earthspeed when account is taken of prevailing winds; so that

$$\frac{E}{\lambda V}$$

is the no. of hours taken to fly one stage.

Substituting these terms, we have

$$n''' = N \cdot \frac{J \cdot E}{365 \cdot H \cdot \lambda V}.$$

We may further note that the total number of hours of flight effected by the service during the year is

$$\frac{N \cdot E}{\lambda V}.$$

We may now consider the following elements of cost:

(i) Depreciation and upkeep of planes. We suppose, first, that technical progress requires the withdrawal of a proportion $1/A$ of the fleet in each year; this entails a total annual cost of

$$\text{(a.1)} \qquad \frac{1}{A}(n' + n'' + n''')P \cdot x_p,$$

where x_p is the purchase-price of a plane, per kg. of weight. There is then the cost of overhaul. If each overhaul cost $1/k$ of the original purchase-price of the plane, the cost of overhaul will be, per hour of flight,

$$\frac{1}{H} \cdot \frac{1}{K} \cdot P \cdot x_p,$$

and the total cost incurred during the year will, therefore, be

$$\text{(a.2)} \qquad \frac{N \cdot E}{\lambda V} \cdot \frac{1}{H \cdot K} \cdot P \cdot x_p.$$

Adding (a.1) and (a.2) together, we have

$$\text{(a)} \qquad P \cdot x_p \cdot \left[\frac{1}{A}(n' + n'' + n''') + \frac{NE}{\lambda V} \cdot \frac{1}{HK}\right].$$

(ii) Depreciation and upkeep of engines. If the working life of an engine be M hours, then, neglecting compound interest, we may assume roughly that $1/M$ of its purchase price must be charged to de-

preciation for each hour it works. The total cost annually incurred under this head will then be

$$\left[\begin{array}{c}\text{Purchase price of engine}\\ \text{power of one plane}\end{array}\right] \times \left[\begin{array}{c}\text{Total no. of hours flown by}\\ \text{the service during a year}\end{array}\right] \cdot \frac{1}{M}$$

$$\text{(b.1)} \qquad = \left[\chi \frac{P \cdot V}{270} \cdot tg \cdot \psi' \cdot x_m\right]\left[\frac{NE}{\lambda V} \cdot \frac{1}{M}\right]$$

where x_m is the purchase price of engine-power, in francs-or per nominal h.p. It will be noticed that, as we have taken the aggregate power needed by a plane and not the power per engine, we need not here consider the number of engines per plane. We next have the aggregate annual cost of overhauling the engines, and this will be given by

$$\left[\begin{array}{c}\text{Cost of sin-}\\ \text{gle overhaul}\end{array}\right] \left[\frac{\text{Total no. of hrs. flown by service per year}}{\text{No. of hrs. an engine works between two overhauls}}\right]$$

$$\text{(b.2)} \qquad = \left[\frac{1}{k'} \cdot \chi \frac{PV}{270} \cdot tg\psi' \cdot x_m\right]\left[\frac{N \cdot E}{\lambda V} \cdot \frac{1}{H'}\right],$$

where k' is the ratio borne by the purchase price of an engine to the cost of one overhaul, and H' is the number of hours an engine may be worked between two successive overhauls. Taking (b.1) and (b.2) together, we then have the total annual cost arising from the engines given by

$$\text{(b)} \qquad \chi \cdot \frac{PV}{270} \cdot tg\psi' \cdot x_m \cdot \frac{NE}{\lambda V}\left[\frac{1}{M} + \frac{1}{k'H'}\right].$$

(iii) Fuel and oil. The aggregate annual cost is evidently given by

$$\left[\begin{array}{c}\text{Consumption of}\\ \text{fuel and oil per}\\ \text{h.p. per hour}\end{array}\right] \times \left[\begin{array}{c}\text{H.p. per}\\ \text{plane}\end{array}\right] \times \left[\begin{array}{c}\text{Total no. of hours}\\ \text{flown by service}\\ \text{per year}\end{array}\right]$$

$$\text{(c)} \qquad = m \cdot \frac{P \cdot V \cdot tg\psi'}{270} \cdot \frac{NE}{\lambda V} \cdot x_c,$$

where x_c is the cost per kg. of fuel and oil.

(iv) Crew. M. Bréguet offers as an approximate formula for cost of crew per hour of flight,

$$(18 + 0.004P) \text{ francs-or}.$$

As total annual cost we have, therefore,

$$\text{(d)} \qquad \frac{NE}{\lambda V}(18 + 0 \cdot 004P).$$

(v) General expenses. M. Bréguet considers that these fall into two parts, one invariable and the other directly proportional to service offered (in tonne-kms.); we have, then, for the total general expenses per annum

$$G + G'\left[N\cdot E\cdot\frac{F_u}{1000}\right], \tag{e}$$

division by 1000 being needed to reduce kgs. to tonnes.

We are now able to find the grand total of charges annually incurred through working a service defined by N, n', n'', (n''' being given as a function of n'). It will be noticed that no separate entry has been made for interest charges: M. Bréguet considers this element as being comprised in the entries made for depreciation and general expenses. Bringing together the five elements (a), (b), (c), (d), (e), we then have the numerator of the fraction which expresses the function of average cost, and which we may now write out in full.

Cost per unit of service offered per annum

$$= \left\{P\cdot x_p\cdot\left[\frac{1}{A}(n'+n''+n''') + \frac{NE}{\lambda V}\cdot\frac{1}{H\cdot k}\right]\right.$$

$$+ \chi\cdot\frac{P\cdot V\cdot tg\cdot\psi'}{270}\cdot x_m\cdot\frac{NE}{\lambda V}\left[\frac{1}{M}+\frac{1}{H'k'}\right] + m\cdot\frac{P\cdot V\cdot tg\cdot\psi'}{270}\cdot\frac{NE}{\lambda V}x_c$$

$$\left. + (18+0.004P)\frac{N\cdot E}{\lambda V} + G + G'\cdot\frac{N\cdot E\cdot F_u}{1000}\right\} \div$$

$$\left\{\frac{N\cdot E}{1000}\cdot aP\left[1-\alpha-\chi\cdot\frac{V}{270}\cdot tg\cdot\psi'\cdot q - \frac{m\cdot tg\cdot\psi'}{300}\cdot\frac{V}{V-v}\cdot E\right.\right.$$

$$\left.\left. - \frac{100}{P} - 0\cdot 03\right]\right\}.$$

We may next substitute the numerical values which M. Bréguet suggests or supposes for the coefficients, other than N, n', n'', and n''', and we then have

$$\text{Cost per unit of service offered} = \frac{200{,}000 + 12{,}500(n'+n''+n''') + 938.1N}{497.76N}.$$

(B) We now turn to the calculation of average and marginal cost. Hitherto, we have been considering the cost of providing a certain service, and not that of carrying so much traffic as offers itself when a given service is being provided; but as our formulas contain no term dependent on the degree of utilization, we must take average cost experienced per unit of traffic per annum to vary inversely with the

amount of traffic secured. The variation of average cost thus occasioned we may call the "hyperbola of utilization," and there will be one such hyperbola for each schedule of service offered. Throughout this kind of range, marginal cost is evidently zero. With this explanation in mind, we return to our formula for average cost of service offered, and consider how average and marginal cost will change as the amount of service offered is changed. To do this, we need first some assumptions about the association between variations of N and those of n' and n''. These assumptions are set out in Table XVII, together with values of n''' calculated according to the formula, $n'''=N\cdot J.E./365\cdot H\cdot\lambda V=N/365$, where the quotient, if fractional, must always be raised to the next whole number. The table begins with figures suggested by M. Bréguet; it then assumes N to increase, and records those points only at which changes in n', n'', or n''' first appear. Thus if N is to rise at all above $1460=4\times365$, then n''' must rise from 4 to 5; we then assume that by increasing the number of stages flown per plane per year, N can for a time be increased without addition to n', but that when N reaches 1601 an increase in n' can no longer be avoided; and so on. We further assume that at $N=2351$ an increase in n'', and an expansion of the establishment leading to an increase in G, both become necessary.

With these figures before us, we can now calculate the course of marginal and average cost of service offered. Values computed for marginal cost throughout, and for the turning points of average cost, are shown in Table XVIII, and the course of average cost is displayed in the figure, in which also the hyperbolas of utilization are illus-

TABLE XVII

PROPOSED VARIATIONS OF N, n', n'', n''', AND G

N	n'	n''	n'''	G
1460	4	2	4	200,000
First at				
1461	4	2	5	200,000
1601	5	2	5	200,000
1826	5	2	6	200,000
1971	6	2	6	200,000
2191	6	2	7	200,000
2351	7	3	7	250,000
2556	7	3	8	250,000
2701	8	3	8	250,000

trated by two examples. There are several considerations that we must bear in mind as we study the figure. (i) For the two hyperbolas, the units of the abscissa are units of traffic actually carried, whereas

for the curve of average cost they are units of service offered. (ii) We cannot regard a point on the hyperbola, and the point vertically below it on the curve of average cost, as marking alternative ways of serving the same demand: the amount of traffic secured will depend in part on the amount and arrangement of service offered, and if the service were reduced some traffic might be lost. (iii) The curve of average cost has been computed on the assumption that changes in the amounts of productive factors used occasion no changes in their several prices. (iv) The curve of average cost is represented by straight lines. Strictly speaking, the downward sloping portions are concave upwards throughout, but the concavity at its largest is very small, being, for instance,

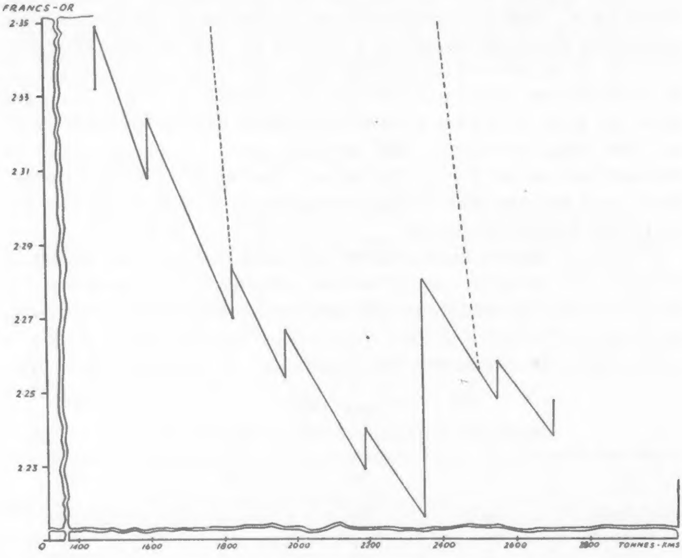

FIGURE 2.—Cost of production per tonne-km. of air transport, at different outputs, from the calculations of M. Bréguet.

approximately 1 in 2,500,000 when $N=1500$. (v) The scale of the figure does not allow the hyperbolas of utilization to appear to the eye as other than straight lines. (vi) The scale, again, must not lead us to exaggerate the changes in average cost; while output is nearly doubled, average cost changes by only about five per cent.

(C) Our last task is to measure the "cost-making pressure" of the different technical and economic coefficients. To avoid the arbitrariness

of the choice of unit, we may use a measure analogous to that of elasticity, and divide proportionate change in cost per unit of service

TABLE XVIII

CHANGES IN MARGINAL AND AVERAGE COST PER UNIT OF SERVICE OFFERED IN AIR TRANSPORT, COMPUTED FROM THE FUNCTION CONSTRUCTED BY M. BRÉGUET

I. *Marginal cost.*

In the intervals within which N is the only parameter to vary, marginal cost is constant and $=1.812$ francs-or per tonne-km.

Where the addition of 1 to N brings a change in n', etc. (e.g. at 1460–1, 1600–1), marginal cost $=26.9$ francs-or per tonne-km., at each point save 2350–1, where it is 151.9 francs-or per tonne-km.

II. *Average cost,* in francs-or per tonne-km.
(Turning points)

N	(AC)	N	(AC)
1460	2.332	2190	2.229
1461	2.349	2191	2.240
1600	2.308	2350	2.216
1601	2.324	2351	2.280
1825	2.270	2555	2.248
1826	2.284	2556	2.258
1970	2.254	2700	2.238
1971	2.267	2701	2.247

offered by attendant proportionate change in the coefficient considered:

$$\kappa_\alpha = \frac{\delta(AC)}{(AC)} \div \frac{\delta\alpha}{\alpha},$$

where (AC) is average cost per unit of service offered. This measure might perhaps be named the *coefficient of costicity*. Since the computation is laborious, we have taken the values of κ for only four coefficients, each at three values of N. In making the calculation, we have to remember that, for instance, not only are (AC) and $\delta(AC)/\delta\alpha$ functions of N, but also (AC) varies with α.[7]

[7] We may note one shortening of computation which is possible where the parameter considered stands in a linear relation to the function, so that its value does not enter into the partial derivative made with respect to it. If we put

$$(AC) = \frac{u}{v}\cdot$$

we have, say,

$$\kappa_{\alpha_1} = \frac{\delta(AC)}{\delta\alpha}\cdot\frac{\alpha_1}{(AC)} = \frac{v_1\dfrac{\delta u}{\delta\alpha} - u_1\dfrac{\delta v}{\delta\alpha}}{v_1^2}\cdot\alpha_1\cdot\frac{v_1}{u_1}\cdot$$

Let us write, and compute, this expression in the form

(7) In the beginning of this report was presented a table taken from a study made by the Warsaw Institute of Economic Research, showing the division of the cost of refining spirits in eight Polish refineries. From the same study[8] we may now take a table showing the relation between level of output and net cost per unit of product through these

TABLE XIX
VALUES OF κ, THE COEFFICIENT OF COSTICITY, FOR FOUR PARAMETERS, AT THREE VALUES OF N

Parameter	Value of parameter	Value of N: 1460	2190	2700
α	0.25	0.5664	0.5621	0.5624
	0.35	1.0928	1.0881	1.0885
	0.45	2.1973	2.1906	2.1924
$tg\cdot\psi'$	0.13	1.1066	0.9805	0.9134
	0.18	1.5767	1.6011	1.5988
	0.23	2.0774	2.7810	2.7867
x_p	9.00	0.1412	0.1037	0.0638
	12.50	0.1859	0.1384	0.0865
	16.00	0.2856	0.1705	0.1081
G'	0.15	0.0657	0.0686	0.0685
	0.20	0.0858	0.0897	0.0894
	0.25	0.1050	0.1097	0.1093

eight refineries (Table XX). In the lower part of the table, the refineries are gathered into four groups; the last group contains the state refineries, which are held to be affected by special factors.

---

(1)
$$\kappa_{\alpha_1} = \alpha_1\left[\frac{1}{v_1 \div \dfrac{\delta v}{\delta\alpha}} - \frac{1}{u_1 \div \dfrac{\delta u}{\delta\alpha}}\right].$$

If, now, we consider a new value of α, α_2, but otherwise do not change (AC), then, on the above assumption of linear relations, we have

$$u_2 = u_1 + \Delta\alpha\cdot\frac{\delta u}{\delta\alpha},$$

$$v_2 = v_1 + \Delta\alpha\cdot\frac{\delta v}{\delta\alpha},$$

where $\Delta\alpha = \alpha_2 - \alpha_1$. Making the necessary changes in (1), we then have

$$\kappa_{\alpha_2} = \alpha_2\left[\frac{1}{v_1 \div \dfrac{\delta v}{\delta\alpha} + \Delta\alpha} - \frac{1}{u_1 \div \dfrac{\delta u}{\delta\alpha} + \Delta\alpha}\right].$$

[8] Reference as in Section II. Tablica XI, XII, p. 7.

(8) In the study by the Warsaw Institute of the cost of production of bar iron,[9] to which, also, reference has already been made, we are given some interesting information concerning the response to changes in output of the various categories of cost, expressed per unit of output. The figures given are taken from the records of one foundry in the second half of 1927 and the whole year 1928. The investigators warn us that the problem is one of great complexity, and that only approximate results can be reached; of the three processes, moreover, blast furnaces, Martin oven, and rolling mill, only the first two proved amenable to this form of analysis. To avoid the difficulty presented by variety in the forms of product, output has been taken to be proportional to the amount of coke consumed. We shall here cite only some of the figures given for the blast furnaces, but in the original source will be found also some figures for the Martin oven.

TABLE XX

RELATION BETWEEN OUTPUT AS PER CENT OF CAPACITY AND NET COST PER UNIT OF OUTPUT, IN EIGHT POLISH SPIRIT REFINERIES IN 1928; FROM *Koszty Oczyszczania Surówki Spirytusowej*, NO. 6 OF THE STUDIES OF THE INSTYTUT BADANIA KONJUNKTUR GOSPODARCZYCH I CEN, 1929

Refinery	Output as per cent of capacity	Net cost per unit of output
1	5	15.63
2	25	13.14
3	28	8.47
4	39	4.85
5	40	11.72
6	46	9.94
7	80	10.06
8	90	10.95
1	5.28	15.63
2 & 3	26.49	10.67
4, 5, & 6	41.28	7.31
7 & 8	85.43	10.39

The following elements of cost are considered: (i) Labor; (ii) Costs general to the whole plant; (iii) Administrative and technical labor; (iv) Costs of administration, excluding salaries; and (v) Power.

(i) Labor. In Table XXI is shown the response to changes in output of the number of worker-hours used per tonne of coke consumed. For the purpose of this table, the smallest recorded consumption of coke per working day, when a ten-hour day is worked and four furnaces are in blast, is set = 100; so, also, is the corresponding number

[9] Reference as for Table XV.

of worker-hours per tonne of coke consumed; and the other figures are expressed as relatives of these bases.

(ii) and (iii) Costs general to the whole plant; administrative and technical labor. The relevant figures are set out in the same table and on the same plan.

(iv) Costs of administration, excluding salaries. This element varies markedly from month to month, because it includes the costs of sale. The investigators assume that it will, on the average, vary in the same way as (ii).

TABLE XXI

RELATION BETWEEN OUTPUT AND INPUT PER UNIT OF OUTPUT OF THREE ELEMENTS OF COST, IN A POLISH BLAST FURNACE, IN SECOND HALF 1927 AND IN 1928

(From *Koszty Produkcji Żelaza Sztabowego*, No. 3 of the studies of the Instytut Badania Konjunktur Gospodarczych i Cen, 1929)

Scheme of work (1)	Output as given by consumption of coke per working day (2)	Number of worker-hours per tonne of coke used (3)	Output, as in (2) (4)	Category (ii), as in (3) (5)	Output, as in (2) (6)	Category (iii), as in (3) (7)
10-hour day with 4 furnaces in blast	100.0	100.0	100.0	100.0	100.0	100.0
	103.0	91.1	111.7	98.6	104.7	88.9
	111.5	86.2	113.3	85.9	110.5	83.3
	114.6	85.8	116.0	88.1	117.6	79.3
	117.5	85.3	127.6	76.5	122.3	75.8
	123.8	80.8	129.3	77.4	132.6	68.1
10-hour day with 3 furnaces in blast	113.7	90.1	132.6	74.7	136.1	62.8
	123.0	82.9	138.6	72.1	145.7	59.7
	128.6	78.7				
8-hour day with 4 furnaces in blast	105.0	84.9				
	107.9	78.1				
	111.1	77.7				

(v) Power. It was found impossible to establish any clear relation between the cost of power and the level of output.

(9) Attention may be called to three studies concerned with the relation between output and the quantities of productive factors used:

"Increasing Return," by G. T. Jones (ed. Clark, Cambridge, 1933), where a historical study is made of the physical quantity of resources absorbed per unit of product, for building, cotton, and pig-iron, in England, and for cotton and pig-iron in the United States.

"Labour and Output in the Coal-Mining Industry in Great Britain," by E. C. Rhodes, D.Sc. (*Journal of the Royal Statistical Society*, xciv, Pt. iv, 1931), which studies the relation between the quantities of different forms of labor employed, and the output of coal, in British mines since the war.

"Production, Output per Head, Prices and Costs in the Iron and Steel Industry, 1924–31," by R. W. B. Clarke, B.A. (*Journal of the Royal Statistical Society*, xcvi, Pt. iv, 1933), in which data of the period studied are used to obtain equations for pig-iron and for steel showing output per head of workers employed as a function of time and of the total production per diem.

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ANNUAL SURVEY OF STATISTICAL TECHNIQUE DEVELOPMENTS IN THE ANALYSIS OF MULTIVARIATE DATA—PART I*

By Paul R. Rider

It would indeed be a difficult undertaking to select from the vast amount of literature on statistical method, which is appearing almost constantly, that material which has particular application in economic theory. Many papers may develop for a given science certain statistical methods which would be equally useful in economics. It is obvious that a large proportion of such methods would probably be overlooked both because of failure to see the papers containing them and because of failure to recognize the applicability of a method to economic science. It would be still more difficult to give a summary of such methods, even those published in journals devoted to economics or economic statistics, which would be comprehensive, systematic, or otherwise adequate. Consequently it has been deemed advisable to limit this discussion to a special phase of the question which seems to be coming into prominence, namely, the desirability in most statistical studies of varying all of the essential conditions simultaneously rather than only one at a time, and the methods that are being devised to cope with this situation.

It has not been unusual to read of the advisability of holding constant all except one of the factors in a statistical complex in order to study its effect more fully. While in some instances this may be desirable it should be realized that much valuable information is contained in the various interactions among all of the forces at work in the complex.[1] This is particularly true in economic and social studies. Fortunately the difficulty or even impossibility of controlling or isolating the various factors involved in such studies can hardly be considered a disadvantage, since it is much more important to observe the effects of these economic forces in their natural setting in which they have free interplay.

Clearly then a study of multivariate statistical populations is highly important, and it is the purpose of this brief article to call attention to some of the recent advances and extensions in the analysis of samples from such populations that have come to the notice of the writer.

* Part II, by Charles F. Roos, will be published in the October issue of this journal. Editor.

[1] See R. A. Fisher, *The Design of Experiments*, Edinburgh and London, 1935, particularly Chapter VI.

In 1928 Wishart[2] derived the simultaneous distribution of the variances and covariances in samples from a normal multivariate population. This was a generalization of Fisher's result for the joint distribution of the two variances and the product moment coefficient from a normal bivariate distribution, the derivation being effected by Fisher's geometric methods. Five years later Wishart and Bartlett[3] used the theory of moment generating functions to deduce the same distribution. The distribution is entirely independent of the sample means. For a sample of n individuals from a p-variate normal population, it is given by

$$\frac{|A_{ij}|^{(n-1)/2}}{\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma\left(\frac{n-i}{2}\right)}\exp\left(-\sum_{i,j=1}^{p}A_{ij}a_{ij}\right)|a_{ij}|^{(n-p-2)/2}da.$$

Here $|A_{ij}|$ is the pth order determinant whose general element is $A_{ij}=n\alpha'_{ij}/2\alpha$, α'_{ij} being the cofactor of α_{ij} in the determinant $\alpha=|\alpha_{ij}|$ of population variances and covariances; α may be defined as the generalized variance of the population. Likewise $|a_{ij}|$ may be defined as the generalized variance of the sample. Its elements are the sample variances and covariances

$$a_{ij}=a_{ji}=\frac{1}{n}\sum_{k=1}^{n}(x_{ik}-\bar{x}_i)(x_{jk}-\bar{x}_j), \qquad (i, j = 1, 2, \cdots, p)$$

in which $x_i=(1/n)\sum_{k=i}^{n} x_{ik}$ is the sample mean of the ith variate x_i for the kth individual. The differential da is the product of the differentials of all the a's.

Hotelling[4] in 1931 established the distribution of a generalization of Student's ratio of the deviation of the mean of a sample from the population mean to the standard deviation of the sample. The square of this generalized ratio may be defined as

$$T^2=\frac{n-1}{|a_{ij}|}\sum_{i,j=1}^{p}a'_{ij}(\bar{x}_i-m_i)(\bar{x}_j-m_j),$$

in which x_i and m_i are the sample mean and population mean respectively of the variate x_i, and a'_{ij} is the cofactor of a_{ij} in the de-

[2] John Wishart, "The Generalised Product Moment Distribution in Samples from a Normal multivariate Population," *Biometrika*, Vol. 20A, 1928, pp. 32–52.

[3] J. Wishart and M. S. Bartlett, "The Generalised Product Moment Distribution in a Normal System," *Proceedings of the Cambridge Philosophical Society*, Vol. 29, 1933, pp. 260–270.

[4] Harold Hotelling, "The Generalization of Student's Ratio," *Annals of Mathematical Statistics*, Vol. 2, 1931, pp. 360–378.

terminant $|a_{ij}|$; T reduces to R. A. Fisher's t when $p=1$. In the following year Wilks[5] gave the distribution of a similar quantity

$$Y = \frac{|a_{ij}|}{|a_{ij} + (\bar{x}_i - m_i)(\bar{x}_j - m_j)|}$$

related to T by the equation $Y=(N-1)/(N-1+T^2)$. Wilks derived the distribution of Y, and incidentally that of T, without certain restrictions postulated in Hotelling's derivation. The quantity Y is distributed in a Beta distribution.

Next in order to be considered after a generalized variance would naturally be the ratio of two independent generalized variances. It will be recalled that this ratio, for the univariate case, plays an important rôle in Fisher's analysis of variance. In fact one-half the natural logarithm of the ratio is Fisher's z. Wilks[6] has succeeded in giving the distribution of the generalized ratio.

Another distribution given by Wilks is that of the ratio of the determinant representing the generalized variance to any of its principal minors. When the order of the principal minor is one less than that of the generalized variance, we have the distribution of the variance of the difference between one variate and its estimate from the regression hyperplane of the remaining $p-1$ variates.

Wilks has also given the distribution of a generalized correlation ratio η, as well as that of a generalization of the expression $1-\eta^2$. (In the case of a single variable the distribution of $1-\eta^2$ can be found from that of η or of η^2 by a simple transformation, but this is not true in the generalizations considered by Wilks.) Both of these generalizations are ratios of determinants.

In a later paper[7] he has developed operational method for finding, directly from the probability law of the sample, the moments of a more general class of statistical functions. The generality consists in the fact that some of the variates may be considered as fixed, the results thus extending to least squares regression problems in which the values of the independent variates are known without sampling error.

From these various distributions Wilks[8] has devised criteria of the

[5] S. S. Wilks, "Certain Generalizations in the Analysis of Variance," *Biometrika*, Vol. 24, 1932, pp. 471–494.

[6] *Biometrika*, Vol. 24, p. 478.

[7] S. S. Wilks, "Moment-Generating Operators for Determinants of Product Moments in Samples from a Normal System," *Annals of Mathematics*, Vol. 35, 1934, pp. 312–340.

[8] S. S. Wilks, "Test Criteria for Statistical Hypotheses Involving Several Variables," *Journal of the American Statistical Association*, Vol. 30, 1935, pp.

Neyman-Pearson likelihood type for testing the following hypotheses which arise in the analysis of normal multivariate data:

(1) That a sample is from a population with specified means.

(2) That two or more samples are from populations having a common system of: (a) means; (b) variances and covariances; (c) means, variances and covariances.

(3) That several sets of variates are mutually independent.

All of the foregoing criteria are "Studentized" functions; that is, they and their probability distribution are completely expressible in terms of the sample observations and do not involve the population parameters.

One of the most important applications of this theory in economics would undoubtedly be to the study of deviations from trend lines and regression curves.

No summary of recent progress in analyzing multivariate data would be complete without reference to Frisch's[9] important contributions, which mention of regressions naturally brings to mind. He has called attention to the danger in regression analysis of obtaining nonsensical results when the variates included in a regression equation contain subsets which are in themselves highly intercorrelated. For example, let us suppose that a large number of observations have been made upon three variates x_1, x_2, x_3, which we may consider to be measured from their respective means and which are connected by two independent linear equations. Each observation could be represented as a point in three-dimensional ($x_1x_2x_3$) space. All of the observation points would lie on a straight line through the origin, and it would be useless, in fact absurd, to try to determine the coefficients of either of the two equations connecting the variates, since a set of points lying on a line which is the intersection of two given planes does not contain sufficient information to determine either plane. In such a case an attempt to determine from the data a regression equation involving three variates would be nonsense.

If nevertheless the attempt were made the regression coefficients would, if the computation were carried out to a sufficient degree of accuracy, turn out to be indeterminate forms (0/0). If errors of observation were present the coefficients would not be exactly of this form but would have a fictitious determinateness created by random errors.

The greater the number of variates included in the analysis, the more complex the situation becomes. We might encounter a whole

549–560; "On the Independence of k sets of Normally Distributed Statistical Variables," ECONOMETRICA, Vol. 3, 1935, pp. 309–326.

[9] Ragnar Frisch, "Statistical Confluence Analysis by Means of Complete Regression Systems," *Universitetets Økonomiske Institutt*, Oslo, 1934.

hierarchy in which some of the variates would comprise a set where a regression equation has a meaning, others forming sets where such equations are meaningless. The study of this hierarchy is what Frisch terms *confluence analysis*. He has developed a technique for dealing with the principal problems of such an analysis for the case of linear regressions.

Suppose we have n variables $x_1, x_2 \ldots, x_n$, ordinarily correlated, attaching to each individual of a population. These variables might for example be the rates of exchange among various currencies. It is natural to inquire whether there exists some more fundamental set of independent variables, perhaps fewer in number than the x's, which determine the values the x's will take. One of the latest writers to consider this problem is Hotelling,[10] who has given an orderly procedure for selecting these new variables in the order of definiteness of their existence or of their importance for the purpose at hand, and of rejecting any which prove to be of little importance, or which are not clearly defined by the data.

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[10] Harold Hotelling, *Analysis of a Complex of Statistical Variables into Principal Components*, Baltimore, 1933. Reprinted from September and October, 1933, issues of *Journal of Educational Psychology*.

A GENERAL DYNAMIC DEMOGRAPHIC SCHEME AND ITS APPLICATION TO ITALY AND THE UNITED STATES

By Silvio Vianelli

1. In the past few years there has been a reawakened interest in studies on the development of population, especially in research for a general law regulating this development. However, the first steps were taken on ground still full of prejudices and old determinisms, so that the results obtained were really not too satisfactory. The formulation of too simple hypotheses and the excessive importance given to considerations and analogies of a pure biological character have led the greater part of those who are interested in the question to the creation of the scheme called the normal logistic.

From the old scheme of Verhulst[1] to the modern one of Pearl and Reed,[2] Yule,[3] Delevsky,[4] and others, all sorts of attempts have been made to compel the various populations, y, to follow in time, t, the function,

$$y = \frac{L}{1 + e^{a_0 + a_1 t}}. \tag{1}$$

Now, to be able to understand the excessive simplicity of the scheme, it is sufficient to observe that (1) not only represents a curve with a first branch of increasing, and a second of decreasing increments, but that is symmetric in the only point of inflection, as the ordinates on the left, y_{b-i}, are equal to the differences, $L - y_{b+i}$, on the right.

Although all of them had acknowledged the effects of outside influences, and particularly of the economic conditions of the develop-

[1] P. F. Verhulst, "Notice sur la loi que la population suit dans son accroissement," *Corréspondance mathématique et physique de l'observatoire de Bruxelles*, T. x, 1838.

Id, "Recherches mathématiques sur la loi d'accroissement de la population," *Mémoires de l'Académie Royale des Sciences, des Lettres, et des Beaux Arts de Belgique*, T. xviii, 1844, and "Deuxième memoire sur la loi d'accroissement de la population," *id.*, T. xx, 1846.

[2] R. Pearl and L. J. Reed, "On the Rate of Growth of the Population of the United States since 1790 and its Mathematical Representation," *Proceedings of the National Academy of Sciences*, Vol. vi, 1920; *Id.*, "On the Mathematical Theory of Populations," *Metron*, Vol. iii, No. 1, 1923; *Id.*, "The Probable Error of Certain Constants of the Population Growth Curves," *American Journal of Hygiene*, May, 1923; R. Pearl; *Studies in Human Biology*, Baltimore, 1924.

[3] G. U. Yule, "The Growth of Population and the Factors Which Control It," *Journal of the Royal Statistical Society*, January, 1925.

[4] J. Delevsky, "Une formulation mathématique de la loi de la population," *Metron*, vii, No. 4, 1928.

ment of population, nobody since Malthus had taken these important factors into explicit consideration.

Pearl and his collaborators have considered this fundamental aspect of demographic equilibrium, but only admitting that at a certain stage of civilization the population of a certain territory cannot surpass a maximum. The fact that those types of functions which become infinite in a finite time have been excluded has certainly represented a scientific gain, but a greater gain might be expected which should have enabled one to derive the law of variation of population from functions (analytically determined) embracing the influence of ambient factors.

Amoroso[5] has tried to bring back the genesis of the logistic curve to a rational field, by stating the conception of demographic elasticity by means of a relation between the logarithmic derivative of the population, considered as a function of time, and the corresponding logarithmic derivative of an index of economic activity.

Using a hypothesis of the analytical form of such elasticity, he has been able to obtain (1).

He has written:

$$\epsilon = \frac{1}{y}\frac{dy}{dt} : \frac{1}{S}\frac{dS}{dt} = \lambda\frac{L - y}{y}$$

where y and S are, respectively, the population and the index of the general welfare, and L represents the maximum of population. Assuming, then, $\frac{1}{S}\frac{dS}{dt} = \mu(y-\alpha)$, and substituting in the preceding, he has obtained

$$\frac{dy}{dt} = h(y - \alpha)(L - y), \qquad h = \lambda\mu,$$

integrating (1). Also, from the point of view of the mechanics of demographic phenomena, we may observe that the hypotheses admitted by the various researchers, about rate by Verhulst, or about acceleration by Delevsky, for example, have led to the creation of schemes of an essentially *cynematic* nature, whereas Amoroso's hypothesis about demographic elasticity has led to the formulation of a really dynamic scheme.

The conception of Amoroso has been enlarged and has had the merit of preparing the way for the construction of a more general dynamic scheme.

[5] L. Amoroso, "L'equazione differenziale del movimento della popolazione," *Rivista Italiana di Statistica*, Aprile 1929.

In fact, Vinci,[6] starting from the stated conception of demographic elasticity and giving such elasticity an analytical form even closer to reality, has obtained in a rational way the so-called generalized logistic curve,

$$y = \frac{L}{1 + e^{-F(t)}}, \tag{2}$$

where $F(t)$ takes a peculiar form which it is generally possible to reduce to a development in series. He has written

$$\epsilon(t) = \frac{1}{y}\frac{dy}{dt} : \frac{1}{S}\frac{dS}{dt} = \phi_1(t)\left(\frac{L}{y} - 1\right),$$

thus indicating that the demographic elasticity, consistently with an unknown function of time, becomes smaller and smaller as the population in its absolute value becomes greater and greater. Also, he has written

$$\frac{1}{S}\frac{dS}{dt} = \phi_2(t)\frac{y}{L},$$

indicating that the relative increase of real incomes is a function, also unknown, of time and of the relative magnitude of population. Substituting the last expression in the preceding, he has obtained

$$\frac{dy}{dt} = y\phi_1(t)\phi_2(t)\left(1 - \frac{y}{L}\right),$$

and integrating (2), for

$$F(t) = \int \phi_1(t)\phi_2(t)dt.$$

Pearl and Reed had already proposed (2) as a generalized form of (1), but they had considered it as an empirical function to be used for an exclusively descriptive purpose in the cases in which it would be impossible to use (1) for $F(t)$ equal to a development in a series.

This general dynamic scheme, since it is based on rational hypotheses, appears more suitable to the changing reality of facts than the abstract scheme of Hotelling[7]; also, it is then possible to pass from

[6] F. Vinci, "La logica della curva logistica," *Rivista Italiana di Statistica*, 1929, and "Ancora sulla curva logistica," *Rivista Italiana di Statistica*, 1930, and *Manuale di Statistica*, Vol. II, Bologna, 1934.

[7] H. Hotelling, "Differential Equations Subject to Error and Population Estimates," *Journal of the American Statistical Association*, September, 1927.—A

a development in a series to the primitive form of $F(t)$, which gives the analytical form of the two components, demographic and economic, of the trend of population in the time and place considered.

If we admit that $F(t)$ might be reduced to a development in a Taylor's series, and if we ignore the powers above the nth degree, (2) gives place to four types of continuous curves, which are generally characterized by several points of maximum and minimum and, therefore, by at least one point of inflection, according to the following:

$$\text{even } n \text{ and} \begin{cases} a_n > 0 \\ a_n < 0 \end{cases} \text{ or odd } n \text{ and} \begin{cases} a_n > 0. \\ a_n < 0. \end{cases}$$

When n is odd and a_n is negative, the curve is asymptotic to the straight line $y=0$ for $t=-\infty$, and to the straight line $y=L$ for $t=+\infty$.

The population would have an increasing trend and this would be the case generally presented by the civilized populations of the present epoch. It has been recognized in this scheme that it is a defect, in comparison with the simple logistic, to augment the abstraction of the whole theory by adding the indeterminateness belonging to the power of the development in series; but we believe that the latter, instead of being a fault, is of the greatest value to the generalized logistic, because the choice of the number of the terms of the polynomial would give the curve the necessary flexibility for adaptation to changing reality, and allow one to obtain the two components, $\phi_1(t)$, $\phi_2(t)$, of which we have spoken, in the way most consonant with truth.

The generalized logistic, besides possessing the great advantage of being asymmetric in comparison with the point of inflection and susceptible of having several points of inflection, would be much more adapted than the simple logistic to the solution of the problem of the sum of several logistic curves by means of a new logistic.

In this brief study, we have tried an application of this general scheme to the available data for Italy and the United States, with the object of estimating its positive significance from a critical point of view.

Of course, owing to the want of exact indexes of the variations in economic conditions of these countries, we have been obliged to limit ourselves to computations which we consider only as first approxima-

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generalization of another nature of (1) has been tried by K. Goldziher, "Contributo alla teria della funzione logistica," *Giornale dell'Istituto Italiano degli Attuari*, Ottobre 1933.—For a critical analysis of these and other attempts, and also of Lotka's theory, see my article, "Evoluzione Economica e Demografica negli schemi delle curve logistiche," *Riv. Ital. di Scienze Economiche*, Maggio-Giugno 1935).

tions, but which may be, nevertheless, interesting from an econometric point of view.

2. The generalized logistic curve under the form expressed by Pearl and Vinci would represent the so-called secular trend of population and it must, to give significant results, be fitted to long periods of time. But, the object of our study being to analyze the mutual influence of the economic and demographic evolution in two of the principal civilized countries, we are led to take into account not only the adequate quantitative data for the two populations but also some economic indexes capable of representing, if not the real value, at least the fluctuations of their economic conditions. Inadequacy of earlier economic data compels one to limit the period analyzed to sixty years.

It must be remembered that, to be able actually to apply Vinci's scheme to a certain population, two series of data are necessary, one representing censuses and a corresponding one for real incomes. The data for the first exist for every civilized country, while the data of the second series can at present be obtained only approximately and indirectly.

The economies of modern states being intimately bound to one another by imports and exports, the yearly volume of foreign trade has been acknowledged as one of the best indexes of the variations of the welfare of these states. Therefore, we have taken foreign trade as an index of real income in Italy and the United States. We have, therefore, summed the annual values of imports and exports of the two countries for the periods 1871–1931 and 1870–1930, respectively, and divided these sums by the corresponding indexes of wholesale prices (basis, 1913 = 100).

The decennial censuses have been considered for the same periods, that is, from 1871 to 1931 for Italy and from 1870 to 1930 for the United States. The Italian population data are all referred to the ancient boundaries.[8] Before proceeding to the elaboration of data, we

[8] The census of 1891 was not made in Italy. The figure has been got by interpolation of a cubic parabola passing through the four points corresponding to the years 1861, 1881, 1901, and 1921. The rough data of the foreign trade and wholesale prices have been drawn from the following sources. For Italy: the values of exports and imports in millions of lire from *Annuario Statistico Italiano;* wholesale prices: for the period 1881–1914, from *Necco's Indexes*, letting 1913 = 100; for the period 1921–31 from *Bachi's Indexes;* for the periods 1871–80 and 1915–20, by calculating by means of weighted means of the prices of a group of goods published by E. Cianci, in *Annali di statistica*, Series vi, Vol. xx. Such calculations for the following years of these periods corresponded almost perfectly with the indexes obtained by Necco and Bachi. For U.S.A.: the values of exports and imports and wholesale prices: for the period 1875–1925 from *Business Cycles and Business Measurements* by Carl Snyder: for the period 1926–1930 from *Monthly Bulletin of Statistics* of the League of Nations; for the period

have smoothed by means of a formula the values given for census counts in Italy and the United States, a process necessary to enable one to fit the general curve without the danger of obtaining results that are unexpected or lacking in concrete significance.

Among the different types of function that could be chosen for the smoothing process, we have preferred the following as giving a good fit,

$$f(t) = a_0 a_1^t a_2^{t^2}, \tag{3}$$

which, when a_2 is close to 1, gives us, for the values of t in an arithmetical progression, values at the function approximately in a geometrical progression.

The fitting of the function to the data, expressed in thousands, by the method of the least squares, has led to the following values:

For Italy $a_0 = 26.801$, $a_1 = 1.06435$, $a_2 = 1.00024$:

For U.S.A. $a_0 = 38.694$, $a_1 = 1.30980$, $a_2 = 0.98692$.

The goodness of fit may be observed in the following tables which give the observed and calculated values:

Population of Italy in Thousands

Years	y_i (observed)	$f(t_i)$ (calculated)
1871	26.801	26.801
1881	28.460	28.532
1891	30.456	30.390
1901	32.475	32.384
1911	34.671	34.525
1921	36.361	36.824
1931	39.537	39.296

Population of U.S.A. in Thousands

Years	y_i (observed)	$f(t_i)$ (calculated)
1870	38.558	38.694
1880	50.156	50.019
1890	62.948	62.977
1900	77.995	77.236
1910	91.972	92.261
1920	105.711	107.345
1930	122.775	121.652

1870–74, having no data for exports and imports and wholesale prices, we have considered as income indexes the production of Snyder (*Business Cycles*, pp. 239 ff.). In calculating the percentage variations of volumes we have joined series drawn from different sources in the following way: We have considered at the same time the last datum of one series and the first of the other, the percentage variation of the last but one datum of the old series has been calculated on the data of this series; the percentage variation of the second datum of the new series has been calculated on the data of this new series.

and is revealed by the quadratic mean deviations between observed and calculated data, resulting, respectively, for Italy and the United States as ± 211 and ± 812. Related to the respective arithmetic means, one has the relative values, $\pm .64$ and ± 1.03 per cent.

3. Given these sixty-year data for the two countries, we have applied the generalized logistic function,

$$y = \frac{L}{1 + e^{-F(t)}},$$

assuming $F(t)$ can be taken as a fifth degree polynomial:

$$F(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5.$$

Calling y_6 the values of the last datum of population for each country (that is to say, the largest) and eliminating from the possible values of L those satisfying the expression and less than y_6, one determines for each constant appearing in the fitted function the following values (obtained from the data in units of one thousand):

For Italy:

$$L = 63.560,$$

$$\begin{aligned} a_0 &= -\,0.31594720, & a_1 &= 0.10781307, \\ a_2 &= 0.00281942, & a_3 &= 0.00019795, \\ a_4 &= -\,0.00000958, & a_5 &= 0.00000249; \end{aligned}$$

and for the United States:

$$L = 159.100,$$

$$\begin{aligned} a_0 &= -\,1.13519660, & a_1 &= 0.35687251, \\ a_2 &= -\,0.00240978, & a_3 &= 0.00106836, \\ a_4 &= -\,0.00004082, & a_5 &= 0.00001043. \end{aligned}$$

If we call S the amount of real income of the population y, and S' and y' their respective derivatives with respect to time, the demographic elasticity, $\epsilon(t)$, is represented by the equation,

$$\epsilon(t) = \phi_1(t)\,\frac{L - y}{y},$$

and the relative variation of real incomes by

$$\frac{S'}{S} = \phi_2(t)\,\frac{y}{L}. \tag{4}$$

Also,

$$F(t) = \int \phi_1(t)\phi_2(t)dt, \text{ and}$$

$$F'(t) = \phi_1(t)\phi_2(t). \tag{5}$$

From (4) we have

$$\phi_2(t) = \frac{S'}{S}\,\frac{L}{y}\cdot$$

From (2),

$$\frac{L}{y} = 1 + e^{-F(t)}.$$

one obtains

$$\phi_2(t) = \frac{S'}{S}\,(1 + e^{-F(t)}), \tag{6}$$

and from (5)

$$\phi_1(t) = \frac{F'(t)}{\phi_2(t)}\cdot \tag{7}$$

Equations (6) and (7) permit calculation of the two unknown functions, $\phi_2(t)$ and $\phi_1(t)$. The demographic elasticity may be transformed into the following form:

$$\epsilon(t) = \phi_1(t)\left(\frac{L}{y} - 1\right) = \phi_1(t)e^{-F(t)}.$$

To obtain the analytic expression of $\phi_2(t)$, we have calculated the relative variations from year to year of the volume of foreign trade, which, ignoring random elements, represents the trend of $\frac{S'}{S}\cdot$ Then multiplying these values by the corresponding ones for $1+e^{-F(t)}$, we have obtained a series of raw data for each country to which a polynomial of the third degree has been fitted by the least squares method.

Figure 1 shows the two series of raw data and $\phi_2(t)$ so obtained for Italy and the United States.

Having calculated the derivative of $F(t)$, the expression $\phi_1(t)$, under the form of a rational fractional function for each of the two countries, is readily obtained from (7).

The results are as follows:

For Italy:

$$\phi_2(t)=8.14583-2.42003t+0.91819t^2-0.11372t^3,$$

$$\phi_1(t)=\frac{0.10781307+0.00563884t+0.00059385t^2-0.00003832t^3-0.00001245t^4}{8.14583-2.42003t+0.91819t^2-0.11372t^3};$$

For U.S.A.:

$$\phi_2(t)=14.61122+8.73138t-5.66159t^2+0.69860t^3,$$

$$\phi_1(t)=\frac{0.35687251-0.00481956t-0.00320508t^2-0.00016328t^3-0.00005215t^4}{14.61122+8.73138t-5.66159t^2+0.69860t^3}.$$

Together with the values of $\phi_2(t)$ and $\phi_1(t)$ are shown those of $\epsilon(t)$ corresponding to the census years and with the central year of each period of ten years.

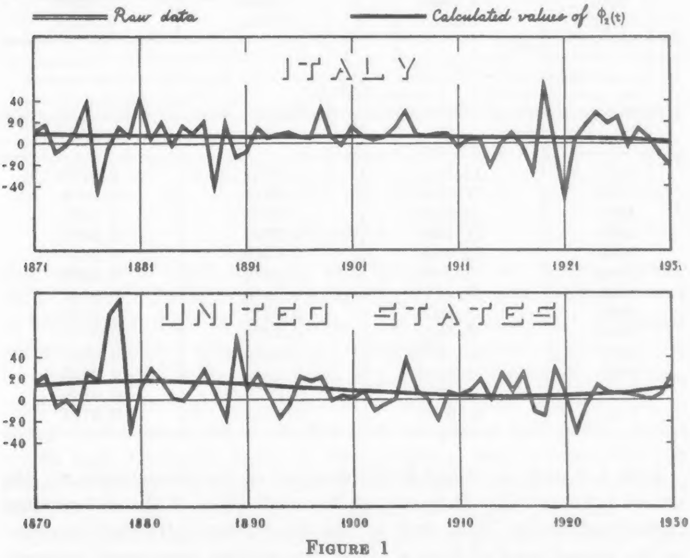

FIGURE 1

From these results the following affirmations may be deduced:

1) The two functions, $\phi_2(t)$ and $\phi_1(t)$, show for each country a trend for which a horizontal straight line cannot be substituted.

2) The demographic elasticity has not diminished in the period examined; on the contrary, the Italian elasticity has shown a slightly accelerating falling off till after the war, then a prominent upward trend.

3) All three functions, $\phi_2(t)$, $\phi_1(t)$, $\epsilon(t)$, have shown general trends which are completely different in the two countries.

ITALY

Years	$\phi_2(t)$	$\phi_1(t)$	$\epsilon(t)$
1871	8.1458	0.0132	0.0182
1876	7.1512	0.0154	0.0199
1881	6.5303	0.0175	0.0214
1886	6.1979	0.0190	0.0220
1891	6.0688	0.0200	0.0218
1896	6.0576	0.0207	0.0213
1901	6.0790	0.0214	0.0207
1906	6.0478	0.0221	0.0199
1911	5.8787	0.0239	0.0201
1916	5.4863	0.0268	0.0209
1921	4.7854	0.0321	0.0233
1926	3.6908	0.0438	0.0294
1931	2.2170	0.0771	0.0476

U.S.A.

Years	$\phi_2(t)$	$\phi_1(t)$	$\epsilon(t)$
1870	14.6112	0.0244	0.0760
1875	17.6488	0.0213	0.0558
1880	18.3796	0.0193	0.0421
1885	17.5885	0.0203	0.0372
1890	15.0164	0.0240	0.0366
1895	11.9669	0.0304	0.0390
1900	8.7133	0.0426	0.0451
1905	5.7656	0.0659	0.0583
1910	3.6617	0.1070	0.0776
1915	2.9617	0.1373	0.0822
1920	4.0534	0.1049	0.0506
1925	6.5968	0.0679	0.0266
1930	14.0799	0.0338	0.0104

It is not easy to establish the complex of factors determining the actual form of $\phi_2(t)$. It is enough to recall some of the determining factors marked by Vinci, such as "the recognition of the right to strike or the punishment of it as a crime, the savings movement, railway, autobus, and air transport development, corporate organization, the tariff policy, inventions and discoveries, etc."

At any rate, one fact seems to be certain, that is to say, on the general trend of this function some circumstances seem to have impinged, for example, wars and other adverse phenomena, in such a way as to restrain the intensity of relative increase of foodstuffs or, more generically, that intensity of relative increase of real incomes

which would have been produced by the increase of the population, during the period examined in both countries.

In fact, the ever-increasing obstacle that the intensity of relative growth of real incomes in Italy has encountered is obvious from the diminishing of values of the function $\phi_2(t)$.

Note that in Italy the value of such a function diminished with notable speed in the period 1871–91, while it was about constant from 1892 to 1907; since then it has once more begun to diminish. The dedecrease in the first period is to be attributed partly to the economic

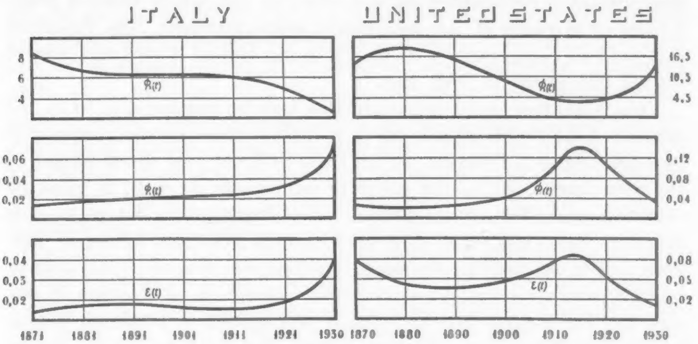

FIGURE 2

depression that Italy suffered, with all the rest of Europe, during 1873–78 and the following years, ignoring a short intermediate period of betterment. To this circumstance must be added, with a smaller weight, of course, the increasing emigration which, depriving the country of a part of the population in an economically productive age, produced an ever greater resistance to the increase of the means of living. It is interesting to observe that the period 1897–1907, during which $\phi_2(t)$ remained more or less constant, was characterized by economic prosperity in our country. At the breaking out of the World War, $\phi_2(t)$ began again its uninterrupted movement of decline.

Certainly the war and its terrible consequences have produced their effects on the secular trend of our function after peace was concluded. The Italian economy, upset by the terrible conflict, was compelled to undergo the crisis of 1921 and, when it was about to recover, the consequences of the universal depression of 1929.

For the United States, the function $\phi_2(t)$ showed an increase in the period before 1882–83, perhaps due to the great immigration, one more increase in the post-war period, and, in general, a strong increase through the greater part of the years analyzed. On the peculiarity of

the movement of this function much could be said, but it is enough to have observed that, during this period, in the case of both Italy and the United States, the intensity of relative increment of incomes has not been proportional to the increase of the population, but a complex of social and technological factors has intervened to limit it.

The function $\phi_1(t)$ has, on the other hand, shown in both countries, and for Italy uninterruptedly, an increasing trend.

The fact, easily observable from Figure 2, that the change in the demographic elasticity in time represents more fully the complex of circumstances represented by $\phi_1(t)$ than of the future relative possibilities of growing of the population, is symptomatic. The functions $\phi_1(t)$ and $\epsilon(t)$ have quite a solidarity in the United States and, except during 1891–1906, in Italy also.

This is another point demonstrating how impossible it is to fit the scheme of the simple logistic curve to reality, for $\phi_1(t)$ must necessarily be admitted as a constant, in contrast with the real development of the phenomena. As to the concrete significance of $\phi_1(t)$, it must be acknowledged that several circumstances intervene to determine this function. Vinci has noted "the institution of abortion, killing of children and of old persons among savage peoples, religious exaltation of celibacy and chastity, the demographic campaign of Fascism in our country, Dr. Stopes' neo-malthusian propaganda in England, migratory policy, the progress of hygiene, etc." And, for instance, the falling of the values of $\phi_1(t)$ and of $\epsilon(t)$ in the United States since 1920, simultaneously with the raising of the values of these functions in Italy, may be largely attributed to the migratory and demographic politics of the two countries. Certainly, ignoring all these particular circumstances, $\phi_1(t)$ must also, however, represent the influence of the way in which the means of subsistence are divided between the growth of population and the amelioration of welfare.

We recall what Amoroso said about his formula of demographic elasticity:

> This formula expresses substantially the fact of daily experience that step by step as the cycle of expansion proceeds to its phase of demographic saturation, lower and lower shares of the increment of subsistence are given to the increment of the population, which signifies that an ever-increasing share is given to the amelioration of life. We note that, having so formulated the principle of *ortesian* action, the error implicit in the first formulation of Malthus is eliminated, according to which all the increment of subsistences appeared given to the increment of population. That error led directly to the sophism of Lassalle's *law of brass.*

The intervals during which $\phi_1(t)$ is decreasing could, therefore, also characterize phases of economic fluctuation particularly favorable to the movement of savings and capitalization, and *vice versa.* The trend of this function and of demographic elasticity is not adapted to an

interpretation of only one meaning; a concrete analysis may lead in this choice.

4. Though we deem the results obtained to be reasonable, we think it fitting to mention some further considerations.

Taking the problem of forecasting by means of the logistic function, it is evident that, though a forecast made by the simple logistic function is difficult and arbitrary, still more irrational and less defensible would it appear using the generalized logistic function *sic et simpliciter*.

We hold to the possibility of making rational prediction, knowing the future movement of the two functions, $\phi_2(t)$ and $\phi_1(t)$. In fact, admitting that the analytical expression of the two functions is reliable also in the future, we could, by means of some simple extrapolations, make exact forecasts. It is a simple hypothesis that finds but little basis in reality, but it teaches us that the extrapolations made with the generalized logistic function, though they are empty of any concrete significance, must give us values which might logically be expected, which will acquire the significance of forecasts only when such hypotheses becomes a reality.

For this, the value of the parameters that appear in the function, though they may not serve to forecast, must not be too far from actual values. In our applications we obtained as values of L for Italy and the United States, respectively, 63.560 and 159.100 (in thousands), values which might be possible upper asymptotic limits of the two populations. As a curiosity, we may add that those two limits would be practically reached by Italy in 2061 and by the United States in 2020. If, therefore, we cannot extract from our functions forecasts that are divinely accurate, at least it is possible to control whether or not for the past one obtains a good fit to the observed data.

The calculations for Italy and the United States are reported in the two following tables, where the extrapolated values and corresponding observed data[9] are shown.

POPULATION OF ITALY IN THOUSANDS

Years	y_i	$f(t_i)$
1811	18.257	17.909
1821	19.000	19.484
1831	21.089	20.812
1841	22.355	22.555
1851	24.162	23.922
1861	25.017	25.174

[9] The empirical data for Italy were published by Travaglini in "La popolazione italiana nel secolo anteriore all'unificazione del Regno," *Annali dell'Universita di Camerino*, Vol. VI, 1932; the American ones by Pearl in *Studies in Human Biology*, *q.v.*

POPULATION OF U.S.A. IN THOUSANDS

Years	y_i	$f(t_i)$
1830	12.866	10.110
1840	17.069	15.002
1850	23.192	21.260
1860	31.443	29.130

One may readily note the very good fit of the function to the Italian data during the period 1811–61, while the extrapolation for the United States during the period 1830–60 would be unsatisfactory.

This signifies that, if the data on foreign trade and reliable indexes of prices for the period 1811–61 were known for Italy, it would be possible to calculate some values of $\phi_2(t)$ and $\phi_1(t)$ very close to those that would be obtained by extrapolating these two functions already arrived at for the period 1871–1931.

We make a last observation. We recall that, in the fitting of the normal logistic function (1), the condition the values y_0 y_1, y_2, must satisfy in order that the logistic should have an upper limit, is given by:

$$y_1^2 > y_0 y_2.$$

Now, if we would adapt the normal logistic curve to the data of Italian population for 1871–1931, putting $y_0 = 26.801$, $y_1 = 32.384$, and $y_2 = 39.296$, this condition not being satisfied,[10] the function would be limitless, that is to say, the upper asymptotic limit of the population would equal infinity.

The fitting of the generalized logistic function to the same data has led instead to a finite superior limit of 63,560,000, as the data satisfy the condition. At present, it is sufficient to have noticed the fact that the fitting of a simple logistic function may give impossible results, since it leads to no significant conclusion, while the fitting of a generalized logistic curve to the same data may be possible.

5. In conclusion, despite the fact that the scarcity of statistical data at our disposal has not allowed a deeper and more comprehensive analysis, the applications to Italian and American data for the periods 1871–1931, 1870–1930, have been sufficiently cogent to persuade us that:

1) The scheme of the generalized logistic curve is, of all the formulas for the growth of population which have been proposed till now, the most adaptable to the changing and complex reality of different countries and times.

2) It can represent adequately the mutual dependence between the

[10] In fact, it would result: $y_1^2 < y_0 y_2$.

economic and demographic evolution of the populations and indicate clearly the action of social force through particular functions.

3) It would permit rational forecasts on the development of population, based on the knowledge of the future behavior of the two functions, $\phi_2(t)$, $\phi_1(t)$.

4) Having proved that the movement of the population is strictly bound to the movement of those two functions, containing a complex of social factors, it is demonstrated that demographic evolution cannot be treated as if it were only or usually a natural biological phenomenon.

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THE MEETING OF THE ECONOMETRIC SOCIETY IN NAMUR, BELGIUM, SEPTEMBER 23 TO 25, 1935

THE FIFTH European meeting of the Econometric Society was held in Namur, Belgium, from September 23 to 25, 1935. The following persons attended: R. G. D. Allen, London; Fernander Banos, Madrid; Hans Bolza, Würzburg; A. L. Bowley, London; Miss M. E. A. Bowley, London; Marck Breit, Warsaw; E. H. Phelps Brown, Oxford; J. J. J. Dalmulder, Rotterdam; P. de Wolff, Amsterdam; L. H. Dupriez, Louvain; Paul Fontigny, Zolder; Ragnar Frisch, Oslo; J. G. Koopmans, The Hague; W. J. de Langen, Wassenaar; G. Lutfalla, Paris; J. Marschak, Oxford; James William Nixon, Geneva; Angelo Della Riccia, Brussels; Paul Rousseaux, Charleroi; Hans Staehle, Geneva; K. H. Stephans, Kiel; J. Tinbergen, Scheveningen; Jan Wisniewski, Warsaw; W. Woytinsky, Paris; Joseph Zagorski, Warsaw; F. Zeuthen, Copenhagen.

MEETING OF THE ECONOMETRIC SOCIETY IN OXFORD, ENGLAND, SEPTEMBER, 1936

THE SIXTH European meeting of the Econometric Society will be held at New College, Oxford, England, during the week-end September 25–29th, 1936. Accommodation with full board will be available in New College from Friday evening, September 25th (dinner), at a charge of 12sh. 6d. per day, inclusive of all gratuities. Members desiring to attend the meeting are requested to write at an early date to Mr. E. H. Phelps Brown, New College, Oxford, enclosing a remittance of 2sh. 6d. They will receive in reply further details regarding programme, accommodation, etc. Drafts or summaries of papers should be sent before August 1st to Mr. J. Marschak, All Souls College, Oxford. In addition to the regular papers there will be organized this year a number of surveys on the present state of econometrics, including economic theory. The main part of the first two days of the meeting will be devoted to these surveys and discussions in connection with them, while the last two days will be reserved for the regular papers. The following is a list of the surveys scheduled: Saturday morning, September 26th: (1) Mr. Keynes' System, R. Harrod, Oxford, J. Meade, Oxford, J. Hicks, Cambridge; Saturday afternoon: (1) Rectilinear Regression, R. D. G. Allen, London; (2) Co-variation, J. Neyman, London; Saturday evening: Colloquium.

Sunday morning, September 27th: (1) Dynamic Equations Underlying Modern Trade Cycle Theories, J. Tinbergen, Rotterdam; (2) International Trade and Payments, A. Lerner, London; Sunday afternoon:

(1) National Incomes (measurement and comparisons), C. Clark, Cambridge; (2) Income Distribution Functions, D. G. Champernowne, Cambridge; Sunday evening: Colloquium.

A list of the regular papers will be sent to those who sign up for the meeting upon request to Mr. Phelps Brown.

The discussion of both papers and surveys would be greatly facilitated if short summaries (not exceeding three typed pages and confined to the main definitions and propositions) could be circulated by the authors at the beginning of the meeting. Duplication of such summaries, sent before September 15th can be done in Oxford (c/o J. Marshak) at the cost of 3/-per page for 50 copies. Authors wishing to use this service should make a remittance when sending their papers.

MEETINGS OF THE ECONOMETRIC SOCIETY IN CHICAGO, ILLINOIS, DECEMBER 28–30, 1936

THE FOURTEENTH American meeting of the Econometric Society will be held in Chicago, Illinois, December 28–30 inclusive. Headquarters will be at the Stevens Hotel, which has quoted special prices for members.

Authors of research papers suitable for inclusion on the program should send manuscripts or summaries to Charles F. Roos, 301 Mining Exchange Bldg., Colorado Springs, Colorado, as soon as possible, and certainly before November 1 when the program will be closed.

In addition to sessions for regular contributed papers, there will be a joint meeting with the American Statistical Association for invited papers, and one or two special symposia.

ERRATUM

In the First Report of the Econometrica Committee on Source Materials, E. H. Phelps Brown, Chairman, (ECONOMETRICA, Vol. 4, No. 2, April, 1936), the formula given on page 128 for the marginal productivity of nitrogen requires correction, a constant having been omitted. It should read

$$\frac{\delta y}{\delta N} = \frac{a_n}{(n+N)^2} \div \left[\frac{a_n}{n+N} + \frac{a_k}{k+K} + c\right]^2.$$

The value of the constant c is not stated in the original paper by Balmukand, but it may be inferred to be approximately 0.074. The insertion of this constant greatly changes the numerical results set out in Table IV (p. 128), and illustrated in Figure 1. The corrected results are set out in the table below, and are illustrated in the corrected Figure.

MARGINAL PRODUCTIVITY OF SULPHATE OF AMMONIA IN POTATO-GROWING, AT DIFFERENT LEVELS OF INPUT OF POTASSIUM, FROM MASKELL'S FORMULA FITTED BY BALMUKAND.

Input of sulphate of ammonia, cwts. per acre	Marginal productivity is expressed in tons potatoes per cwt. sulphate ammonia per acre				
	Input of sulphate of potash, cwts. per acre				
	0	1	2	3	4
0	1.561	1.745	1.819	1.859	1.877
1	0.931	1.069	1.127	1.159	1.175
2	0.618	0.722	0.766	0.790	0.805
3	0.440	0.520	0.554	0.573	0.586
4	0.329	0.392	0.420	0.436	0.445

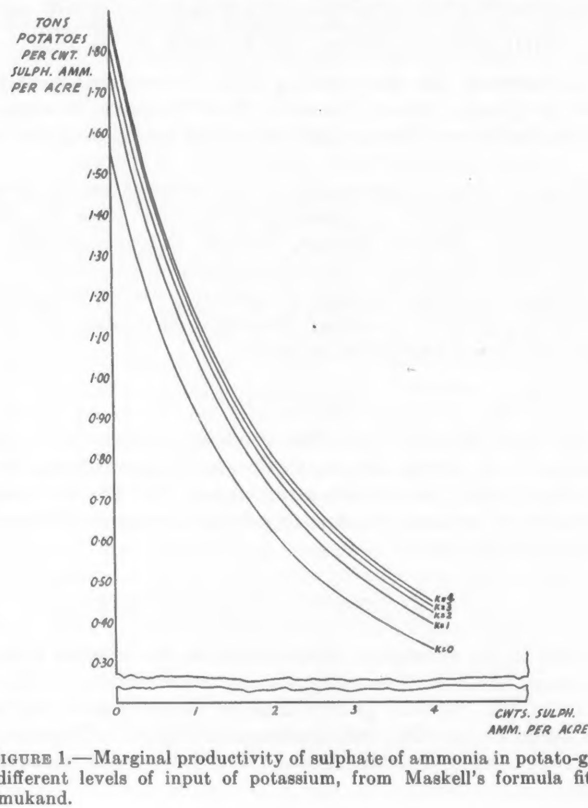

FIGURE 1.—Marginal productivity of sulphate of ammonia in potato-growing, at different levels of input of potassium, from Maskell's formula fitted by Balmukand.

IN MEMORIAM

ECONOMETRICA records with deep regret the death of William Franklin Cram Nelson who had served continuously as assistant editor since the founding of the Journal in 1932. Mr. Nelson was born in Ottawa, Canada, in 1900, and was graduated from Toronto University in 1921. For the last four years he was economist of the Cowles Commission for Research in Economics and lecturer in statistics at Colorado College.

PRICE DATA AND PROBLEMS OF PRICE RESEARCH

By Frederick C. Mills

It is doubtless true in all fields of scientific inquiry that the demands of the speculative theorist have far outrun the lagging pace of observation. The limitations placed upon scientific thought by this deficiency have been especially acute in the social sciences, in which the phenomena studied are usually the resultants of a complex of forces. Under these conditions, verification of hypotheses places heavier demands upon observation. Large numbers of observations taken under varied conditions are needed to support generalizations. Definitive verification is difficult, and conflicts between theories may persist for years. The field of prices, in particular, has been characterized by proliferating theory and sparse observations. This is true even today, when our price data are richer and more comprehensive than they have ever been before. For, though the data are numerous, they remain inadequate to the current needs of economists. In the present survey, some recent accessions of historical data are described and existing materials are briefly reviewed with reference to some of the specific problems with which students of prices are concerned. No attempt is made to cover all the problems that face students of prices. Indeed, such problems are as broad as economic science. Price research is not a restricted field of study; it is merely one line of attack upon the general problems of economics. But the central problems of price research and data bearing on them are of interest to most economists, regardless of theoretical predilections.

I. HISTORICAL STUDIES OF PRICES MOVEMENTS

Students of economic history have long felt that important sources of information concerning economic change were to be found in neglected price records. Scattered studies had shown the value of the records that could be unearthed through intelligently directed historical research. Some seven years ago a comprehensive program directed toward the construction of continuous and well authenticated price series was launched by the International Committee on Price History, under the chairmanship of Sir William Beveridge. A substantial part of the task has now been completed; the major results may be briefly summarized:[1]

Spain.—The researches of Dr. E. J. Hamilton have covered the

[1] I am indebted to Professors Edwin F. Gay and Arthur H. Cole, of Harvard University, for information concerning the present status of the work of the International Committee and activities of associated groups.

period from 1351 to 1815. Two volumes, dealing with the period prior to 1650, have been published;[2] a third volume will carry the record through the Napoleonic era.

Poland.—Studies organized by Dr. Francis J. Bujak have uncovered price records in a number of Polish cities extending from the later middle ages to the outbreak of the World War. Five volumes have already been published;[3] and three more are scheduled to appear in 1936.[4] It is expected that three additional reports, carrying prices in Poznan (Posen), Warszawa (Warsaw) and Lublin down to 1914 will be completed within two years. Two other studies directed by Dr. Bujak, dealing with prices in Danzig from 1540 to 1815, will be published in 1936.

Austria.—Professor Alfred F. Pribram has secured data on commodity prices and wages in Wien (Vienna) and in three other communities from the fifteenth century through the Napoleonic period; these are to be published in 1936.

Germany.—Dr. Ernst Wagemann has carried back to 1792 wholesale price series previously available only to the middle of the 19th century, and has constructed continuous indices for the period 1792–1934.[5]

Investigations by Dr. Moritz J. Elsas have been devoted to the price histories of particular areas, for varying time periods falling between the fourteenth century and 1815. His data, relating to München (Munich), Augsburg, Nürenburg and other communities, will appear in several volumes.[6]

The Netherlands.—Professor N. M. Posthumus has compiled wage and price data for the Low Countries for periods from the fourteenth century to the Napoleonic era. The cities covered include Amsterdam, Utrecht, Anvers (Antwerp), Bruges and Gand (Ghent). Publication is expected in 1937.

[2] Hamilton, E. J., "American Treasure and the Price Revolution in Spain, 1501–1650," *Harvard Econ. Studies*, Vol. XLIII, 1934.

"Money, Wages and Prices in Valencia, Aragon and Navarre, 1351–1500," *Harvard Econ. Studies*, Vol. LI, 1936.

[3] Hoszowski, S., *Prices in Lwów 1701–1914*, Lwów, 1934; Pelc, J., *Prices in Krakow 1369–1600*, Lwów, 1935; Tomaszewski, E., *Prices in Krakow 1601–1795*, Lwów, 1934; Adamczyk, W., *Prices in Lublin from XVI till the end of the XVIIIth century*, Lwów, 1935.

Siegel, S., *Prices in Warszawa 1701–1815*, Lwów, 1936.

[4] Adamczyk, W., *Prices in Warszawa 1540–1700.*

Mika, M. *Prices in Poznan 1490–1815.*

Bieniasz and M. Gorkiewicz, *Prices in Krakow 1796–1914.*

[5] Wagemann, Ernst (ed.); "Die Grosshandelspreise in Deutschland von 1792 bis 1934," Alfred Jacobs and Hans Richter, *Institut für Konjunkturforschung*, Sonderheft 37, 1936.

[6] Dr. Elsas discusses the records from one area in "Price Data from Munich, 1500–1700," *Economic History*, February, 1935.

England.—The English compilations, directed by Sir William Beveridge, include wages and commodity prices for the period prior to 1790. It is expected that the results will appear in four volumes. The first of these will deal with commodity prices from 1530 to 1790; the second with wages for the period prior to and following 1530; the third with prices prior to 1530; the fourth will contain general tables and interpretation.

France.—The major series unearthed in the French studies, under Professor Henri Hauser, cover the period from the middle of the 16th century to the Revolution, with some 15th and early 16th century data. Publication of the report is expected in early 1937.

The United States.—Results of a series of regional studies, dealing with commodity prices in Colonial Pennsylvania, in Boston, the Ohio Valley, South Carolina and New York have already been published.[7] A second volume by Dr. Anne Bezanson, dealing with prices in Pennsylvania from 1784 to 1860, is ready for publication, and further studies of the Philadelphia area will follow. A general summary of the Committee's work in the United States, with inter-regional comparisons, will be contained in a volume by Professor Arthur H. Cole, which will be published in 1936.

The work of the International Committee on Price History and its collaborators covers the period during which the money economy was coming into being and complex price systems were evolving. The materials they have rescued are grist for economists and historians alike, in tracing the genesis of contemporary economic institutions.

II. THE CONSTRUCTION OF PRICE INDEX NUMBERS

Some of the earliest work in prices was directed toward the measurement of alterations in the exchange value of money. This has remained a problem of continuing importance, in spite of the emergence of a diversity of new questions centering about the prices of special cate-

[7] Bezanson, A., R. D. Gray and Miriam Hussey, "Prices in Colonial Pennsylvania, 1720–1775," Wharton School of Finance, *Industrial Research Department Study*, Vol. XXVI, 1935.

Crandall, R., "Wholesale commodity prices in Boston during the 18th century," *Rev. Econ. Statistics*, Vol. XVI, 1934.

Berry, T. S., "Wholesale commodity prices in the Ohio Valley, 1816–1860," *Rev. Econ. Statistics*, Vol. XVII, 1935.

Taylor, G. R., "Wholesale commodity prices at Charleston, S. C., 1732–1791," "Wholesale commodity prices at Charleston, S. C., 1796–1861," *Journal of Econ. and Business History*, Vol. IV, 1931–32.

Stoker, H. M., *Wholesale Prices at New York City, 1720–1800*. Part II, Memoir 142, Cornell Univ. Agric. Exp. Sta., 1932.

Warren, G. F. and F. A. Pearson, *Wholesale Prices in the United States for 135 years, 1797–1932*. Part I, Memoir 142, Cornell Univ. Agric. Exp. Sta. 1932.

gories of goods. Apart, however, from Carl Snyder's interesting effort to develop an index of the general price level, no real attempt has been made to construct a measure purporting to reflect changes in the general purchasing power of money.[8] Recent activities have had more restricted objectives.

Within the sphere of wholesale prices important additions to data and to index numbers have been made. In the United States the index of the U. S. Bureau of Labor Statistics has been steadily improved; it includes at present no less than 784 price quotations. In the United Kingdom the Board of Trade has extended the scope of British price compilations, and has constructed a more comprehensive index of wholesale prices than has been available in the past.[9] Its present index rests upon 258 quotations. The official index of the Dominion of Canada now includes 567 price series. A survey of other national index numbers reveals similar extensions. At the same time weights and technical methods have been generally improved. The economic disturbances of the last two decades have impressed upon makers of index numbers the desirability of employing suitable formulae and of securing broader coverage of commodity markets.

The improvements in the scope and technical accuracy of indexes of wholesale prices have been accompanied by an increase in the number of countries for which such index numbers are constructed. Twenty years ago measurements of this type were available for about 17 countries; today index numbers of wholesale prices (varying in scope and quality, of course) for 41 countries are currently published in the *Monthly Bulletin of Statistics* of the League of Nations. It is notable that in a number of countries official index numbers have replaced private compilations. These developments enable us today to trace price changes over much wider areas and with much greater accuracy than was possible twenty years ago.

Like many another movement of recent years, progress in this field has been nationalistic, with little reference to international requirements. Yet these have been pressing. International price comparisons enter into the classical theory of international trade; theories of purchasing power parity center about such comparisons; recent discussion of currency devaluation and international trade has placed emphasis upon the international incidence of monetary actions by different na-

[8] Cf. "The Meaning and Use of a General Price Index," Gottfried Haberler, *Quarterly Journal of Economics*, May 1928, for a suggestive discussion of the limitations of such an index.

[9] Cf. "The Measurement of Price Changes: Retrospect and Prospect," A. W. Flux, *Journal of the Royal Statistical Society*, Vol. XCVI, Part IV, 1933.

tional governments;[10] international stabilization of currencies, if it is to be effected, will raise questions bearing immediately upon the relations among national price levels and price structures. In the face of the urgent needs of accurate and comparable measurements of price changes in different countries, it is surprising that so little has been done to provide such measurements.

The usual general-purpose index number of wholesale prices does not meet the needs of the student of international movements. More directly aimed at international price comparisons, and far more useful for that purpose, are the series of index numbers constructed by Professor A. L. Bowley.[11] These measurements, similar in composition and in distribution of weights, have been constructed for the United Kingdom, United States, Sweden, Holland, Germany, Belgium, France, Italy, Canada, New Zealand, and South Africa; they are carried forward currently in the *Bulletin* of the London and Cambridge Economic Service. Of value for the same purpose are the series relating to basic materials, quoted in gold, sterling and dollar values, that are published in *The Economist*, of London, and the international comparisons made by the German Institut für Konjunkturforschung. But these various measurements fall short of the requirements of the day, and of the much more severe requirements of a possible program of international stabilization on the basis of a revised gold standard or of cooperative currency management. The carrying out of such a program will require generally acceptable and comparable measurements of changes in various national price levels and in the elements of national price structures that affect international trading relations. Index numbers adapted to these needs are yet to be constructed. There is a task here to which members of an international econometric society may well give attention.

The series of import and export prices brought together by the Economic Intelligence Section of the League of Nations provide useful measurements of the terms of exchange among the various commercial countries of the world. They reveal the tremendous alteration in trading relations in the immediate post-war years, a shift that placed industrial countries in a position of great advantage and colonial areas, producing primary products, at a corresponding disadvantage. This was a phase, of course, of the notable post-war schism between the

[10] Cf. "Exchange Rates and Prices" in *Index*, Svenska Handelsbanken, January 1935, by J. B. Condliffe, for an illuminating account of changes in international price and cost relations in recent years.

[11] "Comparative Price Index-Numbers for Eleven Principal Countries," *London and Cambridge Economic Service*, Special Memorandum No. 24, July 1927.

prices of raw and manufactured goods.[12] In its international aspects, it was reflected in low selling prices of the products of colonial areas and high prices but greatly diminished output with accompanying persistent unemployment, in industrial countries.[13] Depression accentuated this violent break in earlier trading relations.

One perplexing problem arising in the construction of index numbers designed to measure changes in the level of wholesale prices has been definitely faced by the U. S. Bureau of Labor Statistics during the last year. This has to do with changes in the qualities of the commodities quoted. All goods except a rather limited number of standardized primary products are subject to such changes. In general, quality has improved over time. Better steels, better agricultural machinery, better automobiles are made. During a great depression the movement in many lines is in the other direction. The quality of many textile products, for example, is reduced, so that goods may be sold at lower prices. In all these cases the prices that are actually quoted define the resultants of true price changes (changes in the prices of unvarying units of commodity "utility") and quality changes. The maker of index numbers may, of course, restrict his compilations to standardized goods. But in avoiding the Scylla of quality changes he is wrecked by Charybdis. The standardized goods are subject to special price-making forces (as is every distinctive commodity group), and their price movements may not be accepted as typical of wholesale prices in general.

We have here one of the truly insuperable difficulties involved in attempting to measure prices changes over considerable periods of time. However, if it is not possible to take full account of quality changes, it may be possible to reduce the error they cause. The Bureau of Labor Statistics has attempted to do this by securing engineering advice as to the actual degree of improvement in the qualities of certain mechanical equipment included in their general index. Thus it is estimated by engineers, on the basis of field trials, that agricultural machinery in use in the United States in 1932 was 70 per cent more efficient than that in use in 1910–14.[14] Utilizing such studies of the

[12] Cf. *Economic Tendencies in the United States*, National Bureau of Economic Research, New York, 1932. *Aspects of Recent Price Movements*, Bulletin 48, National Bureau of Economic Research, October 3, 1933.

[13] Cf. *Recent Monopolistic Tendencies in Industry and Trade*, Gustav Cassel, League of Nations, Geneva, 1927.

[14] *Report of an Inquiry into Changes in Quality Values of Farm Machines between 1910–14 and 1932*. J. B. Davidson, G. W. McCuen and R. U. Blasingame, published by the American Society of Agricultural Engineers, St. Joseph, Mich., June 1933.

changing efficiency of farm machinery, the U. S. Bureau of Labor Statistics has constructed a revised index of wholesale prices in this field.[15] Similar problems are faced in dealing with motor cars. Measurements of changes in prices per horse-power and per pound avoid some of the difficulties due to improving quality, but no completely satisfactory standard has been attained. At present a committee of industrial experts and economists is collaborating with the Bureau of Labor Statistics in an attempt to improve the price records in this field.

It is proper to say that only a bare beginning has been made in the task of taking quantitative account of quality changes, so that price adjustments may be made for them. The collaboration of economists, merchandising experts and engineers may make it possible to measure price movements with greater accuracy. When all is done that may be done, however, there will remain a great area of market valuations, the area of highly fabricated goods, which will only be covered imperfectly by the best of our index numbers.

In dealing with retail prices and living costs greater problems are encountered. Absence of commodity standardization, narrowness of markets, wide dispersion of quoted prices on similar articles and difficulties in securing weighting factors accurately reflecting consumption habits are some of the obstacles in the way of accurate measurement. But here, also, recent years have brought advances in techniques and in the quality and scope of the records available to economists. Indexes of living costs were available for some 14 countries in 1914; the number increased to 41 in 1935. The records on which such indexes are based have been improved and extended. In the United States a study conducted for the Bureau of Labor Statistics by John H. Cover[16] has provided the basis for an extension of the Bureau's price reporting work, with more exact commodity specifications. In Sweden Dr. Gunnar Myrdal and the staff of the Institute for Social Sciences of the University of Stockholm have compiled detailed records of prices to consumers over the period 1830–1913, and have constructed a cost of living index for the period. By connecting this with the official cost of living index for later years a continuous record of a century of changing living costs was obtained. The published report[17] contains not only the general index, but detailed price records, by counties, for the years 1830–1913. A fruitful comparative study of living costs in Detroit and

[15] "Revised Index of Wholesale Prices of Farm Machinery," Jesse M. Cutts, *Monthly Labor Review*, August 1935.

[16] *The Behavior of Retail Prices.* University of Chicago, 1935.

[17] *The Cost of Living in Sweden, 1830–1930.* Gunnar Myrdal, assisted by Sven Bouvin, London, 1933.

in fourteen European cities was conducted by the International Labour Office.[18] We should note, also, the valuable report on existing cost of living statistics in Canada, France, Germany, Italy, the United Kingdom and the United States, prepared by John Jewkes for the use of an international conference on wage comparisons.[19]

Recent advances in the measurement of living costs have not been restricted to the improvement of price statistics. Progress has been made in securing more accurate and more comprehensive family budget materials, which are essential to proper weighting in the construction of cost of living indexes.[20] In the United States the Bureau of Labor Statistics and the Bureau of Home Economics are collaborating on a comprehensive study of family income and family budgets covering several hundred thousand families and seven occupational groups (wage earners, clerical workers, professional and business salaried workers, entrepreneurs, retired persons, farmers and farm operators.)[21]

III. PRICE-QUANTITY RELATIONS FOR INDIVIDUAL COMMODITIES

In this field the urge from the side of economic theory and the pressure of practical interest have combined to stimulate active research in recent years. The concepts of theory have been sharpened by the studies of Pigou, Moore, Schultz, Vinci, Gilboy, Frisch, Marschak, Amoroso, Leontief, Ezekiel, Ricci, Roos, Working and others, and by controversial exchanges that have defined issues and ideas. Although the compilation and utilization of data have lagged distressingly behind the formulations of the theorists, there has been substantial accomplishment in the realistic study of price-quantity relations for agricultural products. Schultz, Ezekiel, Bean, H. Working and E. J. Working, Warren and Pearson, Waugh and other research workers have defined these relations in quantitative form and have provided measurements of elasticity of demand and flexibility of price for the most important agricultural commodities. For non-agricultural products price-making forces are more complex and analysis is more difficult. Recent work by Roos, in studying the demand for gasoline and residential

[18] *A Contribution to the Study of International Comparisons of Costs of Living.* International Labour Office. London, 1932.

[19] *International Wage Comparisons.* Social Science Research Council Bulletin No. 22, New York, 1932.

[20] Cf. "Family Budgets—Source Materials," by Hans Staehle, ECONOMETRICA, January 1935, for a guide to the more important family budget data at present available.

[21] Cf. "Plans for a Study of Consumption Goods and Services by American Families," Hildegarde Kneeland, Erika H. Schoenberg and Milton Friedman, *Journal of the American Statistical Association,* March 1936.

building, has revealed possibilities of substantial accomplishment in this field.[22]

Progress has been made in clarifying the issues that are faced in dealing with market relations under conditions of imperfect competition, but the difficulties in the way of statistical attack upon these problems are serious. A. J. Nichol has made a study of price leadership in the petroleum industry in the United States, which was a pioneer effort in this field.[23] Highly suggestive leads have been opened by Sraffa, Chamberlin, Harrod, Joan Robinson and others, but little in the way of statistical verification has as yet been found possible. One aspect of the problems that arise in this limbo of semi-competitive trading has been extensively discussed in the United States, within the last two years, in the treatment of basing point prices This old-established marketing method, under which buyers pay the prices quoted at certain central basing points plus transportation from these points to the points of delivery, regardless of whether the goods are actually shipped from the basing points, has been under fire from the Federal Trade Commission and has been investigated by the National Recovery Administration, the Bureau of Business Research at the University of Pittsburgh and, more recently, by a committee of the United States Senate. The publications of these agencies contain some new data on actual prices and transporation rates, as well as discussions of the economic issues involved.[24] The problems discussed transcend the particular controversies relating to the steel industry, for the basing point system of price quotation is employed in other heavy industries in the United States.

In other fields attention has recently been given to price-quantity

[22] Numerous studies of price-quantity relations are listed in bibliographies prepared by the library staff of the U. S. Bureau of Agricultural Economics. See, especially, Bibliography No. 48, *Price Analysis: Selected References on Supply and Demand Curves and Related Subjects*, September 1933 and Bibliography No. 58, *Price Studies of the U. S. Department of Agriculture Showing Demand-Price, Supply-Price, and Price-Production Relationships*, October, 1935. Selected references on the theoretical aspects of the subject are given in a summary by Louise O. Bercaw, in ECONOMETRICA, for October 1934. For examples of demand studies in the non-agricultural field see Charles F. Roos, *Dynamic Economics* (Monographs of the Cowles Commission for Research in Economics, No. 1).

[23] *Partial Monopoly and Price Leadership*. New York, 1930.

[24] Federal Trade Commission, *Report on Price Bases Inquiry, The Basing Point Formula and Cement Prices*, March 1932. *Report of the Federal Trade Commission to the President in Response to Executive Order of May 30, 1934, with Respect To the Basing Point System in the Steel Industry*, November 30, 1934. *Report of the National Recovery Administration on the Operation of the Basing Point System in the Iron and Steel Industry*, November 30, 1934. Ralph J. Watkins and Associates, *The Economics of the Iron and Steel Industry*, New York, 1936.

relations. Merchants who face the direct question as to what effect a given price change will have upon sales have long had reason to question the assumption that price reductions bring increased sales and that price advances diminish sales. Of course, no merchant is able to hold constant the time element and other factors, in order that the precise conditions assumed by economic theorists may be attained. Nevertheless, their experiments in price manipulation and their corresponding observations on sales are decidedly pertinent to the considerations of economists. Some of these experiments, which are discussed by Oswald Knauth in a recent papers,[25] clearly reveal consumer buying habits that differ notably from the finely graduated and rational demand schedules in terms of which much analysis has run. Buying zones for given commodities are the rule, and reduction of prices below those which consumers take to be "proper" for stated articles almost invariably curtails sales, instead of increasing them. Similar observations have been made by many merchandising experts. It is unfortunate that supporting statistical data are meager. The extension of the factual record in this field, in order that the true nature of consumer reaction to price changes may be determined, is one of the tasks before students of prices today.

IV. THE COMPONENT ELEMENTS OF SELLING PRICES: MARGINS AND COSTS

Every commodity price may be looked upon as an aggregate of prices paid for constituent elements of the final commodity—for raw materials, labor, the use of capital, taxes, transportation, distribution, etc. Each of these elements is subject to special price-making forces; their cyclical movements and long-term trends may differ markedly. Knowledge restricted to the change in the price of the composite final product is inadequate to the full interpretation of price fluctuations.[26]

Studies of price margins or differential price relations have represented one attack upon this problem of dividing a final selling price into its elements. Warren and Pearson, in studying variations in distributional margins in New York State, have made important additions to the body of factual data relating to distributional costs.[27] Various investigations of the Federal Trade Commission, in the United

[25] "Some Reflections on Retail Prices," in *Economic Essays in Honor of Wesley Clair Mitchell*, New York, 1934.

[26] This is not to say that causal influences necessarily run forward from the costs of constituent items to the selling price of the final product. Costs must be adapted to conditions imposed by the market.

[27] "Cost of Distributing Food," *Farm Economics*, No. 50, January 1928, and other publications of the New York State College of Agriculture and the Agricultural Experiment Station, Cornell University, Ithaca, New York.

States, have covered similar fields, on a nation-wide basis.[28] This body is at present engaged in a study of the cost of distributing farm products. The Bureau of Agricultural Economics has made some contributions on this same subject, and various private investigations have thrown light on specific aspects of the general problem.[29]

The fruitful possibilities open to studies of differential price behavior have been demonstrated by the work of the Food Research Institute on wheat prices.[30] In Canada the subject of distributional margins was explored by a Royal Commission.[31] There are pitfalls in these fields, of course, since the character of the service represented by distributional margins varies from time to time. When some fabricational operations are also covered by the margin, other problems present themselves, since quality of raw material and degree of fabrication are both subject to change. This is a field of research in which specialized knowledge, relating to the conditions prevailing in specific fields, is essential to reliable results.

A different method of measuring changes in the various elements of cost has been employed by the National Bureau of Economic Research. The U. S. Bureau of the Census, in its biennial census of manufacturing industries, secures statistics, by industries, relating to value of manufactured product, cost of materials, total wages paid and total salaries paid. In addition, though with a somewhat more restricted coverage, the Census tabulates the physical quantities of materials consumed and goods produced by the various industries. By proper adjustment it has been found possible to secure from these records, for a large number of important manufacturing industries, comparable statistics relating to the physical volume of production and total value of product. The latter, in turn, may be subdivided into cost of materials (including semi-finished goods and supplies) and cost of fabrication, including profits. Dividing index numbers relating to each of these cost and value aggregates, in dollars, by the corresponding indices of physical production, one secures measurements of changes in the average per-unit selling price of manufactured goods (i.e., price realized by manufacturer), the average cost of materials and the average cost of fabrication (including profits). Average fabrication cost, per unit,

[28] *Chain Stores: Prices and Margins of Chain and Independent Distributors*, Washington, 1933–34. 5v.

[29] See, for example, *Profits and Losses in Textiles*, S. J. Kennedy, New York, 1936.

[30] For an illuminating discussion of these studies see "Differential Price Behavior as a Subject for Commodity Price Analysis," Holbrook Working, ECONOMETRICA, October 1935.

[31] *Report of the Royal Commission on Price Spreads*, Ottawa, 1935.

may be further subdivided into average labor cost and other fabricational costs (salaries, taxes, interest, profits, etc.), per unit of manufactured product. This procedure has been employed on a comprehensive scale in studying the records of manufacturing industries in the United States for the period from 1899 to 1933. The data are quinquennial for the period 1899–1919, biennial for the period from 1919 to 1933. It has been possible to verify the general accuracy of the results by comparison with independently derived measurements. The movements of the index of average selling price, as thus derived from aggregate records of physical output and value, correspond closely to those of an index derived from quoted prices on manufactured goods. The actual realized prices fall below the quoted prices in periods of severe depression (a natural relationship, because of the under-cover price-cutting prevalent at such times), but the general movements of the two series agree closely.

By this method of indirection, indices of realized per unit prices and of the various components of cost have been derived for the most important manufacturing industries of the United States.[32] (The limitations of the data make it impossible further to subdivide the rather heterogeneous mixture represented by "overhead costs plus profits," but by the use of supplementary materials profits per unit have been distinguished in a provisional way from overhead costs proper.) These measurements throw much light on the economic developments of the past several decades. In particular, they reveal a notable advance in fabricational costs for the post-war period, an advance only partly explained by an increase in the actual contribution of agents of fabrication in these years. Efforts are being made to have census materials compiled in forms better adapted to analysis of this type, and there is promise of further progress in this direction.[33]

These studies fall far short of providing all the information econo-

[32] Following is the record of manufacturing industries covered by these computations:

1899–1914:	35 industries	1923–1927:	62 industries
1914–1923:	52 industries	1927–1933:	113 industries

In addition to the measurements for individual industries, index numbers of realized prices and costs are being constructed for major industrial groups and for all manufacturing industries.

[33] The published results of these studies appear in *Economic Tendencies in the United States*, and in Bulletins 45 and 52 of the National Bureau of Economic Research. A report of a special examination of price and cost changes during the period of operation under the National Industrial Recovery Act is given in Bulletin 56, *Aspects of Manufacturing Operations During Recovery*, May 10, 1935. Further detailed figures will be published by the National Bureau from time to time, as the investigation proceeds.

mists desire concerning changes in production costs. Detailed records of average costs in specific plants are wanted and, of still greater use to economists, records of differential costs—the increments to total costs that may be identified with specific additions to quantities produced. Efforts are being made in the United States, through a recently organized Conference on Price Research,[34] to secure the cooperation of accountants and production engineers in extending such information. In this field troublesome problems in cost analysis complicate the usual difficulties encountered in securing statistics of plant operations, sales and prices. Exploratory studies are now being made in selected industries.

V. THE STRUCTURE OF PRICES

The concept of a price structure as a system of related elements, extending through space and time, has been made more realistic in recent years. Such a concept was, of course, involved in the formulation by Walras of a system of equations defining the conditions of general economic equilibrium. Wesley Mitchell, in his first volume on *Business Cycles* (1913), made the conception more concrete, and defined some of the elements of the price system. Later work has carried this forward.

One phase of the study of the price system has dealt with the characteristics of measurements of price behavior when arranged in the form of frequency distributions. Is varying dispersion of price relatives with the passage of time correlated with the cyclical movements of business, or with other aspects of economic dynamics? What changes occur, with time, in the parameters of frequency distributions of price relatives? Does economic significance attach to such changes? Do distributions of price relatives follow the Gaussian law of error?

These questions have concerned students of prices for some time (witness the writings of Walsh, Edgeworth, Fisher, Bowley, Mitchell, Lucien March, and others), and particular attention has been given to them during the last decade. Some of the more recent contributions will be found in Norman Crump's work on price dispersion,[35] in Maurice Olivier's study of the dispersion of prices in France,[36] in A. L. Bowley's

[34] This Conference, organized in November 1935, includes the Universities of Pennsylvania, Minnesota, Chicago, Columbia, Harvard, the Food Research Institute of Stanford University, The Brookings Institution, the National Bureau of Economic Research, the Central Statistical Board, the Bureau of Labor Statistics and the Bureau of Agricultural Economics. Its purpose is to stimulate and coordinate research work in the field of prices.

[35] "The Interrelation and Distribution of Prices and Their Incidence upon Price Stabilization," *Journal of the Royal Statistical Society*, Vol. 87, Part II, 1924.

[36] *Les Nombres Indices de la Variation des Prix*, Paris, 1927.

discussion of price dispersion and of the significance of the characteristics of distributions of price movements,[37] in Oskar Lange's monograph on price dispersion,[38] and in the publications of the National Bureau of Economic Research.[39]

The events of recent years have drawn attention to the concurrent existence of freely moving prices, of relatively fixed, unchanging prices and rates, and of a wide range of "sticky" prices falling between the two limits. It is a matter of more than passing concern to economists to determine whether the price system is becoming increasingly rigid, whether valorization, public regulation, monopolistic and monopoloid control, trade-marking, and various other modern developments are stabilizing the prices of commodities and services, with a consequent decline in the effectiveness of prices as an instrument for the coordination of economic activities. Data for testing such an hypothesis may be derived from customary price quotations, although a broader coverage, extending beyond the limited range of most price compilations, is needed. Utility rates, the cost of services, wage rates, the prices of highly fabricated goods should all be included in a general survey of price rigidity and flexibility. Having such quotations, measurements other than price relatives of the customary sort are needed to define the aspects of price behavior significant for the purpose. Measurements of price variability, weekly, monthly, or from year to year, throw light on the questions at issue. Even more directly relevant are measures of the frequency of price change. A beginning has been made in the accumulation and analysis of bodies of data bearing on these aspects of price behavior. Measures of the variability of prices of various categories of commodities are given in an early paper by S. J. Chapman and Douglass Knoop,[40] in Abraham Berglund's study of fluctuations in steel prices,[41] in a report by the Federal Trade Commission, of the United States, on *The Grain Trade*,[42] and in the work of

[37] "Relative Changes in Price and Other Index Numbers," *London and Cambridge Economic Service*, Special Memorandum, No. 5, February 1924. "The Action of Economic Forces in Producing Frequency Distributions of Income, Prices, and Other Phenomena: A Suggestion for Study." ECONOMETRICA, October, 1933.

[38] Die Preisdispersion als Mittel zur statistischen Messung wirtschaftlicher Gleichgewichtsstörungen" *Veröffentlichungen der Frankfurter Gesellschaft für Konjunkturforschung*, Neue Folge Heft 4, 1932.

[39] *The Behavior of Prices*, 1927. *Economic Tendencies in the United States*, 1932.

[40] "Dealings in Futures on the Cotton Market," *Journal of the Royal Statistical Society*, Vol. 69, 1906.

[41] "The United States Steel Corporation and Price Stabilization," *Quarterly Journal of Economics*, November, 1923.

[42] Vol. VI. *Prices of Grain and Grain Futures*.

E. G. Peake.[43] A volume issued by the National Bureau of Economic Research[44] contains measurements of variability relating to several hundred commodities and extending over a period of some thirty-five years, and also detailed measurements of the frequency of price change. Similar data appear in a study by Dr. Gerhard Tintner,[45] and use has been made of measurements of the frequency of price change by Gardiner C. Means, in his study of price flexibility.[46] Such measurements, combined, yield a characteristic U-shaped distribution, which reflects the presence of highly flexible and highly rigid prices in the same system. Through the comparison of such distributions, relating to different periods of time, changes in the relative importance of flexible and inflexible prices may be traced.

New light has been shed on the working of the price system by Tintner's study of the patterns of cyclic price movements.[47] Utilizing materials drawn from the trade records of six countries Dr. Tintner has made a detailed analysis of the role of commodity prices in the business expansions and contractions of pre-War experience. The price data employed were those already available from official and private sources in the United Kingdom, France, Germany, Austria, Holland and the United States, but these were subjected to analysis designed to reveal the cyclic movements of the individual price series, freed of random, seasonal and trend elements. Tintner's frame of reference, in his study of price cycles, is a common international scheme, patterned upon that of Thorp and Mitchell,[48] with supplementary use of Spiethoff's[49] standard. Of chief interest, in connection with the present survey, are the detailed measurements of the cyclical price behavior of the individual commodities included in Tintner's study. Here are measurements of frequency of price change, sequence of change (in relation to the reference points of the "typical" cycle), length of cycle movements, length of phases of cyclical advance and decline, amplitude of cyclical movement, and mean monthly rate of change in periods of rise and fall. Some of these, of course, are conditioned by the distinctive procedures of Tintner's study, and their usefulness for other pur-

[43] *An Academic Study of Some Money Market and Other Statistics*, London, 1926.

[44] *The Behavior of Prices*, 1927.

[45] *Prices in the Trade Cycle*. Austrian Institute for Trade Cycle Research, in cooperation with the London School of Economics and Political Science, Wien, 1935.

[46] *Industrial Prices and Their Relative Inflexibility*, Senate Document No., 13, 74th Congress, 1st Session.

[47] *Op. cit.*

[48] *Business Annals*, National Bureau of Economic Research, New York, 1926.

[49] *Handwörterbuch der Staatswissenschaften*, 4th Ed., Vol. VI; article "Krisen."

poses may be somewhat restricted. But in spite of this limitation economists will find in this book a rich body of data, bearing upon many economic problems.

Perhaps most promising, as a means of defining the attributes of the system of prices and determining the relations among its elements, is the application of various principles of classification and the study of the commodity groups thus distinquished. What categories of commodities are marked by distinctive modes of behavior? What principles of classification, possessing significance to the economist, reveal heterogeneities in the price system? Answers to these questions are complicated by the statistical difficulties that arise when the data of time series are subjected to customary tests of significance, but the problems persist. For many years the makers of index numbers have constructed, as incidental by-products, group indices for such classes of commodities as farm products, foods, mineral products, raw materials, manufactured goods, etc. Some of the classifications employed have had a rational basis and have been of great value in interpreting economic changes. The highly useful classifications of the Canadian Dominion Bureau of Statistics, based on "purpose" (distinguishing consumers' and producers' goods), on degree of fabrication and on origin, as well as on character of component materials, may be cited here. Various other groupings have been of little use, revealing no persistent and significant differences among the categories compared. Not enough use has been made of principles of classification bearing directly on questions of concern to economists. But already there has accumulated a great deal of material on the price behavior of commodity groups, still largely awaiting exploitation in the study of the elements of the price system.

Steady improvements are being made in the accuracy and significance of group index numbers. The United States Bureau of Labor Statistics has in process a comprehensive program of checking and revising various group index numbers relating to retail markets and living costs as well as to wholesale markets. An index of retail food prices, with weights based on retail sales, is being constructed and a similar index for clothing is contemplated; improved index numbers of rents, with sub-divisions for different rent levels, and of gas and electricity prices are under construction. These will be elements of a new general-purpose retail price index, comparable to the general wholesale price index and to current indices of production, employment, etc. Indices for commodity groups at wholesale are undergoing similar revision, with technically qualified representatives of industry being called upon to cooperate with economists and statisticians.

The National Bureau of Economic Research has applied to the

commodities for which price quotations are compiled by the U. S. Bureau of Labor Statistics grouping principles based on origin, purpose, degree of fabrication, character of market, degree of control over short period supply, relative importance of wages in the total value of product, character of cyclical behavior of prices and other considerations.[50] These classifications are being utilized and tested in connection with the continuing work of the National Bureau in this field.

The system of prices involves a geographical dimension, as well as temporal and functional relations. At a given moment, as Zapoleon[51] and Working[52] have demonstrated, there are differences in the absolute levels of quotations on identical commodities, in geographically separated markets. These differences may be sufficiently systematic as to permit the construction of isotims, or lines of equal prices, corresponding to the isotherms of the meteorologist. Freight rates, tariffs, differences in competitive conditions, variations between surplus and deficit areas and other factors help to account for the differences that are found in a study of the regional structure of prices. There are, moreover, regional differences in price behavior in respect of variability, frequency of change and trend over time. The cyclical movements of commodity prices show pronounced national divergencies from a common pattern. Such national differences are doubtless to be expected. Geographical dispersion is unexpectedly large, also, within such an economic system as that of the United States or Canada. Even within such a country as Sweden Myrdal's study of living costs[53] reveals surprisingly large price differences from county to county.

Here, as in treating so many of the other problems of price research touched upon in preceding pages, the chief requirement of the economist is more data, selected with reference to issues of economic importance. Just as general measurements are inadequate, if not faulty, as representatives of a complex economic situation, so price quotations and price index numbers that purport to represent conditions over a wide area, marked by heterogeneous conditions, are grossly inadequate and misleading. In the development of the synthetic tools—averages, index numbers, coefficients of correlation—that enable the statistician to summarize a great variety of observations in single representative

[50] Index numbers derived from some of these classifications have been published in *Economic Tendencies*, pp. 584–8, and in *Bulletins* 40, 42, 48, 53. Later measurements will appear in *Prices in Recession and Recovery*, to be published in 1936.

[51] L. B. Zapoleon, *Geography of Wheat Prices*, U. S. Department of Agriculture Bulletin, No. 594.

[52] *Factors Determining the Price of Potatoes in St. Paul and Minneapolis*, Technical Bulletin No. 10, University of Minnesota, Agricultural Experiment Station.

[53] *Op. cit.*

figures, undue emphasis has perhaps been laid on the central tendencies, the uniformities, the persistent relations, that may be discovered among apparently discrepant movements. But generalizations based on these apparent uniformities, taking no account of differences, divergencies and aberrations, may be subject to very large errors indeed. Realism in economic research calls for attention to the behavior of smaller groups, more homogeneous elements of the grand total. Perhaps in no field of economic interest is this more important than in that of prices.

VI. THE DEFLATION OF COST, PRICE AND VALUE SERIES

Among the important uses to which price index numbers are put is that of making transitions from the money level to the "real" level of activity—to the level of goods and services. The problem appears in a wide variety of circumstances. It is, in one form, the problem of "deflating" the prices received by any economic group for goods sold or services rendered by prices paid for the goods or services bought by that group. Thus an index of hourly wage rates may be corrected by an index of living costs for the purpose of measuring changes in "real" wage rates, that is, in the volume of goods and services for which an hour of work may be exchanged. Or an index of the prices received by farmers for the commodities they sell may be "deflated" by an index of the prices of goods bought by farmers, for the purpose of measuring changes in the per unit purchasing power of farm products. Again, from relative changes in average import and export prices, for a given country, we may seek to derive indices of the barter terms of trade between that country and all those with which it does business. In another form the problem arises in reducing an aggregate value figure to its equivalent in physical terms. Measurements of changes in the aggregate wage bill of a country, accurately corrected for changes in living costs, become measurements of changes in the real income of the wage-earning population. Figures defining changes in the aggregate money income of farmers, corrected for changes in the prices of goods bought by farmers, become indexes of changes in the aggregate real income of the farming population. In slightly different form the problem emerges when total imports or exports, in dollars, are to be reduced to quantum terms, or when the aggregate values of building contracts awarded are to be used in estimating changes in the physical volume of building.

If we were able to make such transitions accurately for cost, price and value data relating to all important economic groups, if we had a complete arsenal of "deflating" index numbers appropriate to all the needs of the economist, we should be on far firmer ground than we now are in attempting to interpret the processes of economic life. We could

then measure with accuracy alterations in the terms on which goods and services are exchanged among various producing and consuming groups. Changes in the real income of economic groups, in the volume of consumable goods their money would buy, could be defined with precision. But here, also, our requirements far outstrip our resources. It is true that we have passed beyond the day when men considered an index of wholesale prices an appropriate deflator for all price and value series, but the list of specialized index numbers suitable for effecing transitions from the money level to the goods level in the numerous specific cases that arise in economic research is yet a scanty one.

A great advance has been made in recent years in the construction of index numbers of living costs for industrial workers. The coverage of such index numbers is wider, the weights used are more accurate and the technical procedures employed are vastly superior to those in use a short two decades ago. Little has been done, however, to construct similar measurements for groups other than industrial wage earners. The index of prices of goods bought by farmers in the United States is an approach to such a measure, but not all elements in farmers' living costs are included.

Progress in this field will be facilitated by the development of measurements of the type discussed in the preceding section, relating to changes in various elements of the structure of prices. Indeed, the matters discussed in this section and the one preceding are aspects of the same problem. When we have defined the elements of the price system, when we are able accurately to trace price changes affecting all distinctive elements of that structure, then the way will have been prepared for moving with precision from the level of monetary valuations to the level of goods and services. What can be done by direct attack on the measurement of shifts in the physical volume of production and trade and in the internal flow of goods and services must be done, of course. But much will remain to be done by indirection, using price indices as a bridge to the level of physical phenomena. For this purpose accurate knowledge of the details of price movements is essential.

VII. SUMMARY: SOME LIMITATIONS OF THE STATISTICAL RECORD

In spite of the advances of the last two decades in refining the tools of price analysis and improving the records of price changes, serious limitations persist. Perhaps first among these are defects of coverage. It has been generally true that compilations of prices have begun with those that are most readily available—prices of the great agricultural staples, of the raw materials of industry, of simple manufactured goods. Available records of the prices of services, of capital goods, of more highly fabricated consumers' goods have been meager. Generalizations

about price movements have been based, in the main, on samples that are not truly representative of quoted prices of economic goods, in the broad sense. If the system of prices were homogeneous, if all elements behaved in much the same way under the stimulus of monetary and industrial changes, the use of readily available quotations would not lead to serious error. But it is now clear that the price system is highly heterogeneous, that the prices of different categories of goods and services are marked by distinctive modes of behavior. The extension of our records to areas not now adequately covered is an urgent requirement today.

In making such an extension research workers will face in intensified form the problem of quality changes. In some degree this may be cared for through technical evaluation of such changes. But what may be done in this direction is limited. All commodities except a restricted number of staples are subject to quality changes that may not be reduced to quantitative form. The errors thus introduced into the measurement of price changes are not great over short periods; they may be considerable over periods of a decade or more. This fact reenforces the conclusion, to which one is led by other considerations, that the measurement of long-period price movements is subject to errors of great magnitude.

Reference has been made to the need of developing price indices for economically significant commodity groups. The heterogeneous nature of the price system, with wide diveristy of behavior in different categories of goods, calls for such specialized index numbers, if accuracy of measurement is to be secured. Indices for special categories of goods are needed, also, for deflating purposes, for reducing incomes, prices and rates to "real" terms. Various hypotheses concerning causal sequences in business cycles and in other economic movements emphasize the role of price changes among particular classes of commodities —organic and inorganic goods, capital goods, consumers' and producers' goods, raw and manufactured goods. The testing of these theories calls for the development of specialized price measurements. Finally, these group measurements are needed in the study of price relations, in seeking to define the structure and attributes of the system upon which we still depend, in the main, for the coordination of economic activities.

Economists have as yet made little progress in the realistic study of costs and of the relation of costs to selling prices. A major reason for this is found in the inadequacy of the available data. Costs are not matters of public record, as are many prices. Indeed, even the cost records of private business fail to meet the needs of the economist, particularly in respect of the differential costs that are emphasized in

economic reasoning. In this lack of knowledge of the cost schedules of producers, and of the actual marketing experience of industrialists, is found one reason for the weakness of economic science in the face of the pressing issues of the day. The establishment of working relations with accountants, engineers and marketing experts, in order that their records may be made to answer the questions of concern to economists, is a necessary step if economics is to be made a more realistic science.

Finally, we should note the need of non-price series coordinate with our price records. We study price changes for the light they throw on the working attributes of contemporary economic systems. But price movements are significant in relation to changes in volume of production, stocks of goods, volume of imports and exports, exchange rates, employment, wage payments, profits, saving and capital formation, and in other elements of a functioning economy. This, of course, is appreciated in the compilation of these records, but too little attention has been given to securing fully comparable records covering these different aspects of economic activity. Production, wage and employment series covering exactly the same industries as those to which a given price index relates provide us with intelligence of much greater value in economic analysis than would a series of indices relating to different industries, and only roughly comparable. A beginning has been made in the formulation of fruitful hypotheses and the building up of accurate and comparable measurements for the study of economic change. The betterment of price records must go hand in hand with the improvement of measures defining changes in other elements of the economic system.

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THE PROFIT-EXPERIENCE OF PRODUCERS AND THEIR RESPONSE TO PRICE

THIRD REPORT OF THE ECONOMETRICA COMMITTEE ON SOURCE MATERIALS FOR QUANTITATIVE PRODUCTION STUDIES[1]

By E. H. Phelps Brown
Chairman of the Committee

(A) When we turn from the costs of the single firm to the conditions of supply related to a product, or group of cognate products, in a region as a whole, then a functional treatment ceases to be applicable, except perhaps in the longest run. For more proximate relations, we must turn now to the study of the firm: it is here that we may view the processes by which are created the potentialities of supply existing at a given moment in a given trade.

Such studies are represented in agriculture by the surveys which record the results obtained in different sizes of farm, or by different methods of cultivation. For retailing and banking, the reader will call to mind the studies of Professor Secrist of Northwestern University.[2] The National Bureau of Economic Research also has recently published an important study of *Industrial Profits in the United States*, by Professor Ralph C. Epstein.

We may here offer illustrations of three further inquiries: in the first two, the positions of a number of firms are considered simultaneously; in the third, the story of a group of firms is followed through time.

(1) Table XXII presents evidence submitted by Mr. W. H. Coates, LL.B., B.Sc., to the British Committee on National Debt and Taxation, in December 1925.[3] Mr. Coates compiled these data from mate-

[1] See the Introduction on p. 123 in the April issue of Econometrica.

[2] *Competition in the Retail Distribution of Clothing—a Study of Expense or "Supply" Curves*, Chicago, 1923; *Expense Levels in Retailing—a Study of the "Representative Firm" and of "Bulk-Line" Costs in the Distribution of Clothing*, Chicago, 1924; *The Widening Retail Market and Consumers' Buying Habits*, Chicago, 1926; *Banking Standards under the Federal Reserve System—a Study of Norms, Trends, and Correlations of the Assets, Deposits, Expenses, and Earnings of Member Banks*, Chicago, 1928; *Margins, Expenses and Profits in Retail Hardware Stores—Studies Determining the Relationships of Margins, Expenses, and Profits to Volume of Sales and City Size, and Measuring their Regressive Tendencies*, Chicago, 1928; *Banking Ratios—a Study of the Operating Results of Member Banks with Special Reference to the Twelfth Federal Reserve District, and to California*, Stanford University, 1930; *The Triumph of Mediocrity*, Chicago, 1933.

[3] The material is now taken from the *Report of the Committee on Industry and Trade*, the Balfour Report, *Factors in Industrial and Commercial Efficiency*, London, H. M. Stationary Office, 1927, I, 460.

rials accessible to him as Director of Statistics and Intelligence in the Department of Inland Revenue. The figures for each group are based upon a fairly large random sample of corporate enterprises, and all figures shown for a given year are based upon the trader's annual account made up to a date within that year, which is the "Income Tax Year," running from April 6 in one calendar year to April 5 in the next. The classification of concerns has been made according to the descriptions of their trade given by the taxpayers in making their return, though some concerns are not confined to one class of trade alone. The result studied is profit as percentage of turnover, where profit is taken as defined for the purposes of Income Tax and has been subjected to the deduction allowed by the Tax Commissioners for wear and tear or depreciation of plant and machinery, and where turnover consists in total receipts from sale of goods or services, but does not include receipts from property owned by the taxpayer but not used by him in his business.

(2) From the report of the Warszawa (Warsaw) Institute on *Some Problems of the Coal Industry*,[4] are reproduced the figures of Figure 3. These figures display the findings of an inquiry into the economies of mechanization in the mines of the Krakow-Dombrova district. In each figure the unit of the base line represents a single mine, not a quantity of coal handled, and the mines are arranged in descending order of the cost therein studied, so that a given mine may occupy different places in the orders of different figures.

The reader may be reminded of the similar diagrams presented by Professor Taussig in his paper "Price-fixing as Seen by a Price-fixer," in the *Quarterly Journal of Economics*, XXXIII, 1918–19.

(3) In 1856 an act of the British Parliament made limited liability available to every company complying with the requirements of registration, and every registered company was required to make an annual return. The files thus accumulated provide a record of the formation and dissolution of companies, though the record yields itself only to most laborious research. This task has been undertaken by Mr. H. A. Shannon and also by Prof. D. H. Macgregor,[5] and a part of their findings is summarized in Table XXIII and in Figure 4. The reader will not

[4] *Aktualne Zagadnienia w Przemyśle Wglowym*, No. 2 of the studies of the Instytut Badania Konjunktur Gospodarczych i Cen Warszawa (Warsaw), 1929.

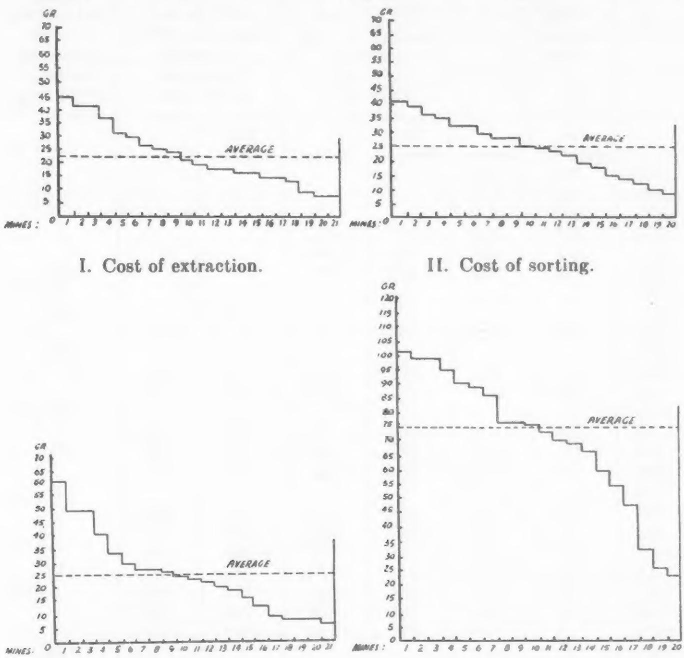
[5] Shannon, "The First Five Thousand Limited Companies and their Duration," *Economic History*, supplement to *Economic Journal*, Jan. 1932; "The Limited Companies of 1866–83," *Economic History Review*, IV, 1932–34. Macgregor, "Joint Stock Companies and the Risk Factor." *Economic Journal* XXXIX, 1929. Reference may also be made to Shannon, "The Coming of General Limited Liability," *Economic History*, Jan. 1931, and to Macgregor, *Enterprise, Purpose, and Profit*, London, 1934.

TABLE XXII

TABLE PREPARED BY MR. W. H. COATES, SHOWING PERCENTAGE OF TOTAL TURNOVER EXAMINED IN EACH OF SEVEN BRANCHES OF PRODUCTION FALLING WITHIN EACH OF TWENTY-FIVE GRADES OF PROFIT ON TURNOVER, IN GREAT BRITAIN, 1912–13 AND 1922–23

Profit as per cent of turnover	Cotton		Wool		Iron & Steel &c.		Miscellaneous metal industries		Food		Wholesale distribution		Retail distribution		Total	
	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23	1912–13	1922–23
−20 and below	—	1.88	—	0.81	0.08	1.87	0.06	1.93	—	0.05	0.01	0.22	—	0.11	0.03	0.87
−19.9 to −10	—	5.15	—	1.42	0.08	4.63	0.13	3.25	—	0.33	—	0.58	—	0.24	0.03	2.07
− 9.9 to − 5	—	7.10	—	0.95	0.17	5.42	—	2.09	—	0.76	—	0.58	—	1.20	0.04	2.39
− 4.9 to − 0	1.16	16.15	1.10	6.09	1.95	5.04	1.17	14.54	—	10.17	3.31	9.03	—	3.81	1.92	8.96
0.0 to 0.9	0.85	8.27	0.48	3.29	2.63	1.72	21.28	3.56	2.46	6.98	16.32	14.77	1.80	5.31	9.44	8.33
1 to 1.9	2.40	7.51	0.66	1.71	2.58	0.92	10.61	5.63	16.83	8.23	10.78	18.18	—	7.33	7.69	9.86
2 to 2.9	2.86	6.95	5.31	7.24	4.86	3.70	5.66	11.70	5.29	13.15	20.51	11.98	7.80	17.48	11.02	10.25
3 to 3.9	8.08	6.62	5.59	4.65	18.47	1.50	9.02	5.74	18.70	6.50	16.73	8.82	7.32	9.15	14.57	6.45
4 to 4.9	9.45	4.39	6.33	3.53	3.98	9.56	11.34	7.87	9.21	5.23	10.75	7.76	7.34	7.18	8.69	7.41
5 to 5.9	19.60	7.32	12.85	5.35	13.52	8.64	9.01	4.18	20.91	9.06	5.85	5.26	13.91	17.20	11.45	7.48
6 to 6.9	10.10	2.85	4.51	3.36	6.99	3.85	4.88	3.59	1.84	3.20	3.45	8.04	19.83	7.04	6.22	5.52
7 to 7.9	10.36	3.93	7.79	6.02	9.34	6.60	0.98	3.46	17.23	2.85	0.86	2.32	6.44	4.76	5.86	3.85
8 to 8.9	11.20	3.85	9.19	3.54	7.53	5.44	3.00	3.49	2.41	3.61	1.55	1.65	—	5.28	3.96	3.40
9 to 9.9	9.29	2.98	15.70	3.18	3.83	6.75	3.20	2.87	1.56	2.06	2.74	1.01	4.35	2.24	4.01	2.85
10 to 10.9	6.35	1.80	10.02	6.24	3.30	5.73	0.70	3.48	0.95	3.00	0.82	1.62	2.32	1.68	2.23	2.98
11 to 11.9	3.02	1.05	7.85	4.46	2.53	1.50	2.94	2.82	—	2.91	2.02	0.99	5.55	2.69	2.59	1.75
12 to 12.9	0.79	1.14	6.28	7.82	4.60	1.67	0.78	2.16	—	1.26	2.71	0.40	6.99	1.02	2.95	1.29
13 to 13.9	1.29	1.27	1.96	2.74	2.25	2.00	1.41	1.18	—	1.53	0.53	1.18	5.59	0.76	1.48	1.41
14 to 14.9	0.53	1.40	0.90	5.47	1.60	1.38	0.01	1.23	—	0.68	—	0.49	1.46	1.35	0.54	1.08
15 to 19.9	2.05	4.58	3.48	10.76	6.18	7.92	10.58	9.11	0.20	11.23	0.62	1.89	3.57	3.00	3.24	5.47
20 to 24.9	0.62	3.28	—	3.71	1.23	8.02	2.45	3.16	2.04	3.46	—	1.32	3.67	0.64	1.10	3.31
25 to 29.9	—	0.42	—	4.85	0.82	3.04	0.07	2.04	—	1.63	—	0.74	—	0.35	0.18	1.51
30 to 39.9	—	0.07	—	2.10	1.44	2.37	0.56	0.26	—	1.31	0.15	0.49	2.06	0.12	0.59	0.93
40 to 49.9	—	0.04	—	0.49	—	0.58	0.12	0.43	0.37	0.81	—	0.18	—	0.02	0.05	0.33
50 and over	—	—	—	0.22	0.04	0.15	0.04	0.23	—	—	0.29	0.50	—	0.04	0.12	0.25
Total	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00	100.00

wish to use this material without referring to the original papers, in which he will find illuminating commentary and more detailed analysis of the data. Yet the present figures must not be cited even by way of example without a reminder that the company there studied is a legal entity, whose formation and dissolution do not necessarily mark the beginning and end of a productive activity; thus, a company may have



I. Cost of extraction.

II. Cost of sorting.

III. Cost of loading in trucks.

IV. Total cost of extraction, sorting, and loading.

FIGURE 3.—Variation by mines in three elements in the cost of coal, from a study by the Warzawa (Warsaw) Institute of 21 mines in the Krakow-Dombrova district of Poland, 1928. Costs are given in grosze (=0.01 zloty) per tonne.

been formed because a private trader hoped to transfer from himself to his creditors the losses he saw coming, and a company may have been dissolved because the concern was bought by new money entering a trade in boom. Only a detailed study of the figures in relation with their legal and economic setting can reveal their true significance. Their interest, however, is so apparent that the type of study which they represent must not pass unmentioned here.

TABLE XXIII

SURVIVALS OF COMPANIES REGISTERED IN LONDON WITH LIABILITY LIMITED BY SHARES, FROM FIGURES RECORDED BY H. A. SHANNON (A, B, C) AND D. H. MACGREGOR, (D)

Year	A Companies formed 1856–65 and existing in 1865: remainder existing at given date		B Companies formed 1866–74 and existing in 1874: remainder existing at given date		C Companies formed 1875–83 and existing in 1883: remainder existing at given date		D Companies formed and existing in 1880: remainder existing at given date
	Industrial	General	Industrial	General	Industrial	General	
1865	579	1,134					
1868	441	823					
1871	339	632					
1874	295	503	1,178	1,650			
1877	266	438	900	1,214			
1880	238	385	670	916			780
1881							720
1882							636
1883	209	346	557	733	954	2,001	
1885							456
1886	178	329	473	604	763	1,501	
1889	166	287	404	525	618	1,192	
1890							333
1892	148	259	364	467	534	1,004	
1895	138	241	335	409	468	861	274
1898	117	214	299	361	387	745	
1900							233
1901	102	198	274	325	333	664	
1904	91	186	260	294	288	603	
1907	86	173	248	277	265	551	
1910	84	165	244	264	244	509	173
1913	83	157	235	250	232	469	
1916	80	154	221	238	219	438	
1919	76	145	217	233	202	426	
1920							145
1922	62	132	164	217	160	378	
1925	61	127	155	210	154	349	
1928	58	124	149	200	141	331	
1929							126

Note to columns A, B, C.—Only companies formed primarily for work in the home market are included. Very small companies, and companies for gas, water, local halls, and clubs, are excluded.

General companies comprise: lead, etc., mines; quarries; lead manufactures; clay, bricks, cement; omnibuses, trams; household goods; food and provisions; breweries, flour mills; hotels, restaurants; land and buildings; theatres; newspapers; paper and printing; farming accessories; petty money-lending; insurance; financing; banking; mercantile and trading; electrical.

Industrial companies comprise: coal and iron mines; iron and steel and general engineering; iron and steel products; specialized engineering; railway rolling stock; shipbuiding; coal by-products; shipping; cables and telegraphy; textiles; chemicals.

(B) We now proceed to a discussion of two topics, the supply-curve of effort and the influence of price on the outputs chosen by entre-

preneurs. The two topics differ in kind, yet the second may comprise the first. They differ because the first takes no account of time—a price once posited is supposed indefinitely maintained; whereas, in the reaction of the entrepreneur to a given price, the second topic com-

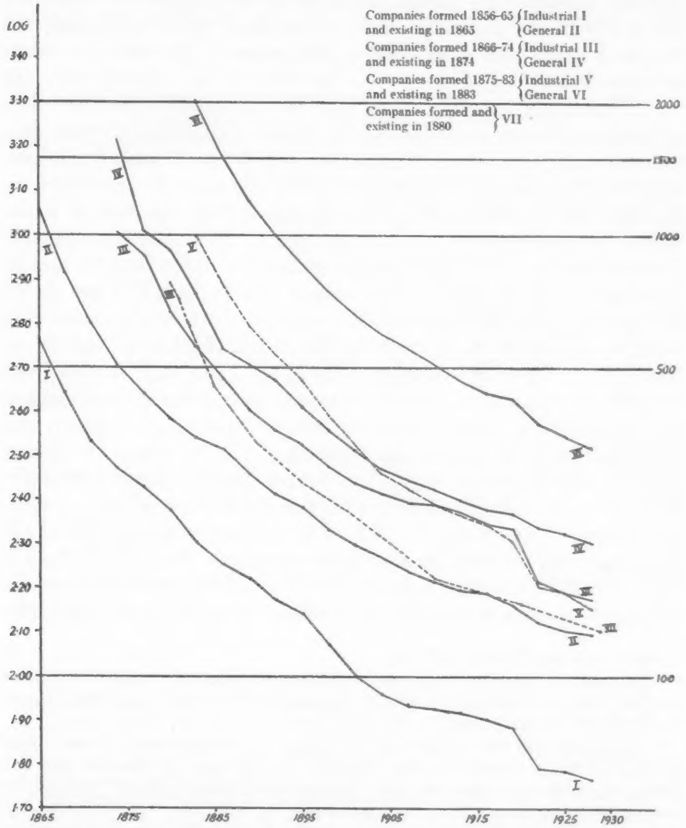


FIGURE 4.—Survival curves (logarithmic) of seven groups of British Limited Companies, from the data of Mr. Shannon and Professor Macgregor.

prises his guesses at the costs and prices of the future. They differ also because the first regards real costs and real rewards alone, whereas the second is concerned principally with reckonings in money. Yet here appears the overlapping, for, in so far as the decision to change

output changes the work required of the man who decides, the second topic includes the first. Apart, however, from the smallholding, there can be few units of enterprise in which this inclusion is important. It follows, then, that in studying the second topic, the statistician may without inconsistency adjust the price of the product to show, not the power of the unit of product to command goods in general, but the ratio which that price bears to the prices of alternative products or of productive factors.

(1) The supply-curve of effort may be studied where we have data of piece-rate wages and of worker's output. An example of such data may be taken from the evidence given before the (British Parliamentary) Select Committee on Coal of 1873;[6] the boom in the heavy industries that followed the Franco-Prussian War resulted in a rise within two years of more than 50 per cent in the money piece-rates of some miners in the British coalfields, and it is interesting to inquire what effects we can find in their output.[7] In Table XXIV are shown the figures presented by three witnesses,[8] all of whom were concerned with the management of mines in Northumberland and Durham; in Table XXV are shown relatives derived from the original data,[9] and enabling us to compare the rise in money piece-rates with changes in the proportion of possible working days utilized by the worker, and with changes in his output during the days he did work. The figures concern only the hewer, that is, the worker who cuts the coal from the face and whose task is ended with the loading of the coal into the tubs; the evidence is therefore not affected by the contemporary changes in the proportion of other kinds of labor employed. From Table XXIV, again, it will be seen that between 1871 and 1873 output was increased only in the Pease's West Collieries, so that, unless the good seams were

[6] *Parliamentary Papers*, 1873, x.

[7] The relation between wage and output in the British coalfields has been studied over a longer period by Dr. Rhodes in the paper mentioned above, *Journ. Roy. Stat. Soc.*, XCIV, Pt. IV, 1931.

[8] The figures for the thirty-seven collieries in Northumberland were given by George Baker Forster, mining engineer and manager of three or four collieries, Q. 3015 *seq.*; those for the Adelaide's and Pease's West Collieries, by Joseph Whitwell Pease M.P., colliery-owner, Q. 4221 *seq.*; those for the 157 collieries in County Durham, by Lindsay Wood, mining engineer and colliery-owner, Q. 3398 *seq.*

[9] Thus column (3) expresses the quotient of average earnings per day worked divided by average output per day worked, which itself is derived from Table XXIV by dividing, where necessary, the figures of its column (8) by those of its column (6), expressed per year; column (4) expresses the quotient of column (6) in Table XXIV divided by column (7) in that table; column (5) has already been filled in obtaining the figures of (3); and column (6) is the product of (4) and (5) in the same table.

at this time being rapidly exhausted, we have no allowance to make for inferior working conditions. It is true that at the same time more hewers (save again in Pease's West) were being employed, and the new men may at first have been inferior to the old; but in the Northumberland group the increase was less than 7½ per cent, and in the Durham group it was less than 6 per cent. There remain, however, three other

TABLE XXIV

DATA CONCERNING THE PAYMENT AND OUTPUT OF HEWERS IN CERTAIN COLLIERIES OF DURHAM AND NORTHUMBERLAND, 1870–73, FROM EVIDENCE BEFORE THE SELECT COMMITTEE ON COAL OF 1873

(1) Mines	(2) Year	(3) No. of hewers	(4) Total output, tons	(5) Average earnings per hewer per day worked	(6) Average no. of days worked: per hewer	(7) Average no. of days worked: per mine	(8) Average output in tons per hewer
					per fortnight		per year
37 collieries in Northumberland	1871	7,249	5,774,834	65 d.	8.85	10.3	796
	1872	7,435	5,770,428	96d.	8.93	10.3	776
	1873*	7,784	5,306,508	109¾d.	8.01	9.4	681
					per year		per day
The Adelaide's Colliery in S. Durham	1870	133	194,000	50d.	258	291	5.85
	1871	128	182,000	51d.	258.02	300.5	5.77
	1872	113	164,000	79½d.	244.83	292	5.70
Pease's West Collieries in S. Durham	1870	863	860,000	49½d.†	268.1†	298	4.68
	1872	1,035	975,000	75½d.	264.3	302	4.24
					per fortnight		per year
157 collieries in County Durham	1871	14,664	16,090,107	56d.	9.2	10.91	1097
	1872	15,127	15,382,875	78d.	9.1	11.03	1016
	1873*	15,552	14,696,761	93d.	9.01	10.90	940

* Based on figures of first quarter only.

† In the original record given as for 1871, but evidently in error.

factors which may disturb the reckoning. In the first place, three witnesses report (Q. 1969, 6509, 6768–9) that the miners combined to keep output down, with the explicit purpose of keeping wages up: "since we had begun to accumulate a little stock, the men have resolved to reduce the work for another day in the fortnight." None of these witnesses, it is true, spoke for the districts we are studying, but in so far as the same consideration influenced the miners there, the

relation between the price and the output of effort is obscured. At a time of boom in the industry, however, such a consideration can have had little effect on practice, and we have, indeed, much evidence (though as it happens, only for other districts) that the miners chose shorter hours for their own sake. In the second place, the miners may have sought shorter hours only as they sought higher wages, as part of the trade union struggle for a better standard of life; but here again (with the same qualification) we have much evidence that the higher wages

TABLE XXV

FIGURES DERIVED FROM TABLE XXIV AND EXPRESSED IN RELATIVE FORM (1872 = 100)

(1) Mines	(2) Year	(3) Piece-rate wage	(4) Average no. of days worked per hewer as percentage of days mines worked	(5) Average output per hewer: per day he worked	(6) Average output per hewer: per day mine worked	(7) Average output per hewer: per year
374 collieries in Northumberland	1871	66.2	99.1	103.5	102.6	102.6
	1872	100.0	100.0	100.0	100.0	100.0
	1873*	116.8	98.3	97.8	96.2	87.6
The Adelaide's Colliery in South Durham	1870	61.4	105.8	102.6	108.5	108.2
	1871	63.3	102.4	101.2	103.6	106.7
	1872	100.0	100.0	100.0	100.0	100.0
Pease's West Collieries in South Durham	1870	59.4	102.2	110.4	112.8	112.0
	1872	100.0	100.0	100.0	100.0	100.0
157 collieries in County Durham	1871	67.9	102.2	106.8	109.1	108.0
	1872	100.0	100.0	100.0	100.0	100.0
	1873*	127.6	100.2	93.4	93.6	92.5

* Based on figures of first quarter only.

were the effective inducement. Alexander Macdonald, the miners' leader, himself stated that there had been a movement for shorter hours, independent of the rise in wages, but "of course it has spread since the advance in wages" (Q. 4566–73). Third, there is the fact that the immediate cause of the reduction of the working day in 1873 was the coming into force of legislation reducing the hours of work of the boys, who hauled the coal, and upon whom, therefore, the hewer was dependent. There is, however, no reason to suppose that the consequent reduction was not acceptable to the men: G. B. Forster, speaking for Northumberland, said that "the boys are only worked fifty-four hours a week, and both the men and masters considered that it would be

better to work five days of ten hours than six days of nine hours" (Q. 3036–7).

While, therefore, we by no means have here a pure experiment, we can safely attribute a large part of the decrease in output to the influence on the hewer of higher piece-rates. If then, with reservations in mind, we turn to the figures, we find that in Northumberland, including the effect of the less number of days the mines worked, an increase between 1871 and the first quarter of 1873 of some 76 per cent in the money piece-rate was accompanied by a decrease of more than 14 per cent in the output per hewer; and in Durham, supposing that the reduction in the number of days the mines worked was not due to the wishes of the miners, we find that nearly the same decrease in output was associated with an increase of nearly 88 per cent in the money piece-rate. Of the two mines for which separate figures are given, only the Adelaide's Colliery allows of direct comparison with the two county groups, and between 1871 and 1872 this mine shows in money piece-rates a rise greater than, and in output a decrease intermediate between, those of the two groups. It may also be observed that we seem here to meet the backward-sloping supply curve; when the price of coal rises, and wages follow, these collieries produce less than before.

It is interesting to add that the reduction in working hours seems to have been accompanied by an increase in output per hour. We are told that in 1871, 1872, and 1873, the hewers of Durham were actually at the coal face for $6\frac{1}{2}$, $5\frac{3}{4}$, and $5\frac{1}{2}$ hours respectively (Q. 3454), while for Northumberland the corresponding hours were $6\frac{1}{2}$–7, 6–$6\frac{1}{2}$, and $5\frac{1}{2}$–6, (Q. 3036–7) Fixing the latter series at $6\frac{3}{4}$, $6\frac{1}{4}$, and $5\frac{3}{4}$, and comparing the hours with the output per hewer per day the mine worked, we obtain the following indexes of output per hour:

	Northumberland	*Durham*
1871	95.8	94.5
1872	100.0	100.0
1873	106.3	97.6

(2.1) The influence of price on the output undertaken by entrepreneurs has been very fully studied for certain branches of agriculture in the United States, especially by the economists of the U. S. Department of Agriculture. This kind of study is illustrated from the work of Dr. Louis H. Bean by Figure 5, here reproduced from a mimeographed report of an address on "Characteristics of Agricultural Supply and Demand Curves," delivered by Dr. Bean before Section K of the American Association for the Advancement of Science, June 1932. The data used are illustrated by the figures for potatoes given in

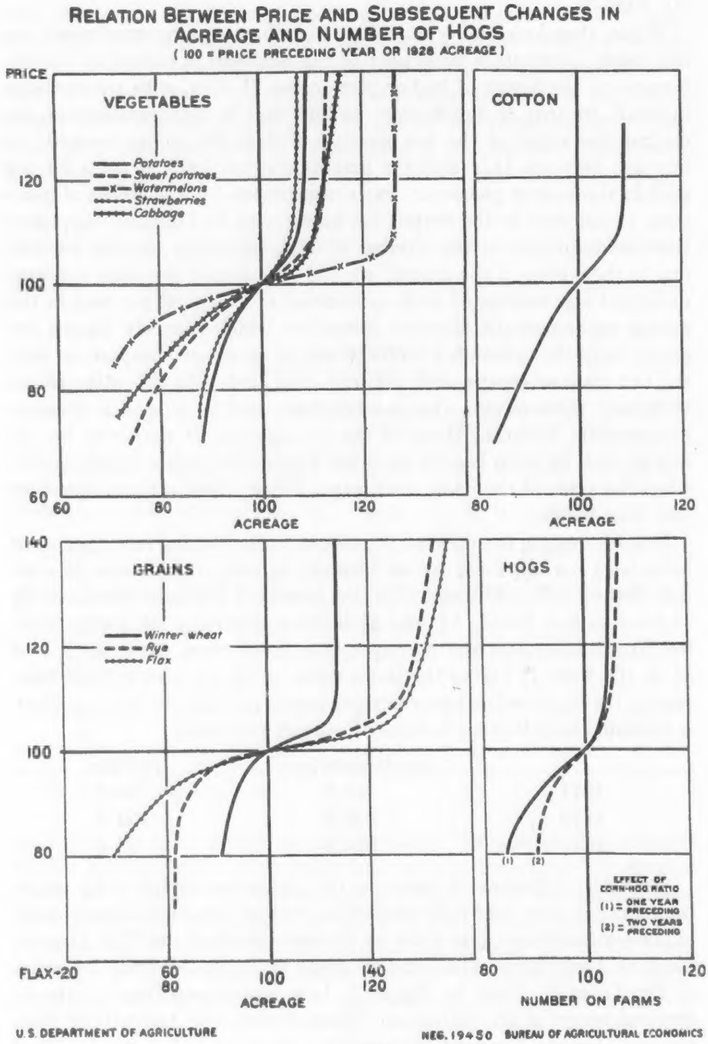

RELATION BETWEEN PRICE AND SUBSEQUENT CHANGES IN ACREAGE AND NUMBER OF HOGS
(100 = PRICE PRECEDING YEAR OR 1928 ACREAGE)
PRICE
VEGETABLES
Potatoes
Sweet potatoes
Watermelons
Strawberries
Cabbage
COTTON
ACREAGE
ACREAGE
GRAINS
Winter wheat
Rye
Flax
HOGS
EFFECT OF CORN-HOG RATIO
(1) = ONE YEAR PRECEDING
(2) = TWO YEARS PRECEDING
FLAX-20
ACREAGE
NUMBER ON FARMS
U.S. DEPARTMENT OF AGRICULTURE
NEG. 19450 BUREAU OF AGRICULTURAL ECONOMICS

FIGURE 5

Table XXVI.[10] For each crop the dollar price is regarded as only a first step towards obtaining the quantity which is significant for the

TABLE XXVI

RELATION OF PRICE OF POTATOES RECEIVED BY PRODUCERS TO CHANGES IN THE UNITED STATES POTATO ACREAGE, 1919–1930

Year	Acreage	Acreage change	Weighted average farm price per bushel*	Index of farm prices July–June†	Farm prices adjusted, 1927–28 = 100	Effect of price: One year preceding	Effect of price: Two years preceding	Trend	Residuals
	1,000 acres	*1,000 acres*	*Cents*		*Cents*	*1,000 acres*	*1,000 acres*	*1,000 acres*	*1,000 acres*
1919			223.8	220	140.3				
1920	3,657		131.5	152	119.4				
1921	3,941	+284	121.3	119	140.6	+345	+ 50	−100	− 11
1922	4,307	+366	73.9	130	78.4	+365	+ 40	−100	+ 61
1923	3,816	−491	94.2	132	98.5	−420	+ 50	−100	− 21
1924	3,310	−506	76.5	142	74.4	− 70	−325	−100	− 11
1925	3,074	−236	183.5	143	177.1	−450	0	0	+214
1926	3,120	+ 46	140.8	129	150.6	+375	−350	0	+ 21
1927	3,476	+356	108.4	138	108.4	+365	+ 50	0	− 59
1928	3,837	+361	61.3	137	61.7	+280	+ 50	0	+ 31
1929	3,338	−499	136.2	133	141.3	−490	+ 25	0	− 34
1930	3,394	+ 56	97.3	98	137.0				

* United States Department of Agriculture Yearbook 1930, p. 775.

† Index numbers of farm prices received by producers (August 1909–July 1914 = 100).

TABLE XXVII

INDEXES OF CORRELATION BETWEEN ACTUAL CHANGES IN ACREAGE (AND IN NUMBER OF HOGS), AND CHANGES ESTIMATED LARGELY FROM PRICE; FROM THE STUDIES OF DR. LOUIS H. BEAN

Product	Index of correlation	Index corrected for numbers of observations and constants*
Potatoes, U. S.	0.985	0.874
Sweet potatoes	0.960	0.616
Cabbage	0.966	0.907
Strawberries	0.993	0.977
Watermelons	0.984	0.956
Flax	0.995	0.961
Rye	0.996	0.967
Cotton	0.986	0.968
Hogs	0.967	0.901

* Corrections made according to the formula given by M. J. B. Ezekiel in his paper on Applications of the Theory of Error to Multiple Curvilinear Conclusions (*Proceedings of the American Statistical Association*, December 1928). The numbers of observations and assumed constants were respectively: potatoes, 8 and 7; sweet potatoes, 8 and 7; cabbage, 8 and 5; watermelons, 8 and 5; strawberries, 10 and 7; wheat, 8 and 6; flax and rye, 8 and 7; cotton, 9 and 5; hogs, 9 and 7.

farmer's decision; thus for wheat, potatoes, cabbage, cotton, and rye, the price is divided by an index of the prices of farm products generally; the prices of flax and of sweet potatoes are divided by those of their

[10] Here taken from a mimeographed report by the U. S. Department of Agriculture of addresses delivered by Dr. Bean at meetings of the Pacific Northwest Potato Committee in February 1931.

respective rivals in the fields, dark Northern spring wheat, and cotton; the price of hogs is divided by that of the main food of hogs, corn. The figures thus obtained are submitted to the method of "graphic curvilinear correlation."[11] From the regression thus obtained for each crop, the adjusted price is found that is associated with no change in output, and in the diagrams both this and the corresponding crop are set = 100, other figures being expressed as relatives to these as base. It will be noticed that the curve for flax is drawn on a scale half as great as

TABLE XXVIII

ACREAGE, PRICES, AND PURCHASING POWER OF WHEAT IN THE UNITED KINGDOM, 1924–33

(Purchasing power is expressed in terms of all agricultural produce)

Year	Price of wheat per cwt. Sept.–Nov.		Price index numbers Sept.–Nov. 1911–13 = 100		Purchasing power of wheat	Acreage of wheat in in the U.K	
			Wheat	All agricultural produce		Year (June)	1,000 acres
	s.	d.					
1924	12	5	175	162	108	1925	1,552
1925	11	3	158	154	103	1926	1,652
1926	11	8	164	150	109	1927	1,708
1927	10	6	148	140	105	1928	1,459
1928	9	6	134	141	95	1929	1,385
1929	9	4	131	145	90	1930	1,405
1930	6	11	97	133	73	1931	1,250
1931	5	9	81	115	70	1932	1,343
1932	9	10*	138	102	135	1933	1,744

* Approximate price given under the Wheat Quota scheme.

that for the other curves in the same diagram. The curves show, in general, a much greater elasticity for small than for large variations in the adjusted price, but elasticity is greater for large downward than for large upward deviations. The significance of these relations is tested when the outputs deduced from them are compared with the outputs actually realised: the coefficients of correlation thus obtained are set out in Table XXVII.[12]

(2.2) It is interesting to compare the American curve for wheat with a similar curve obtained for Great Britain by Dr. Keith Murray,[13]

[11] Described by Dr. Bean in the *Journal of the American Statistical Association*, December 1929, and December 1930.

[12] Here taken from Dr. Bean's paper, "The Farmers' Response to Price," in the *Journal of Farm Economics* XI, 3, July 1929.

[13] Here taken from *The Farm Economist* I, 4, October 1933 published by the Agricultural Economics Research Institute, Oxford, England.

(Table XXVIII and Figure 6). While a 10 per cent rise in the American adjusted price above the level of maintained output is associated with an increase in output of less than 5 per cent, and a fall of 10 per cent in the adjusted price with a decrease of 20 per cent in output, the

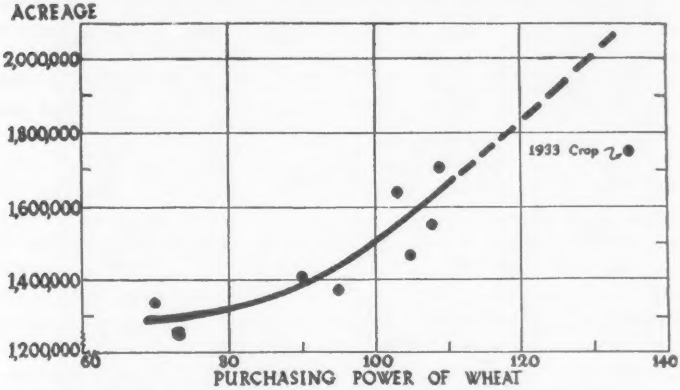


FIGURE 6.—The relationship of the purchasing power of wheat (in terms of all agricultural produce) in September–November to the acreage of the succeeding crop in the United Kingdom, 1924–33, from the study by Dr. Keith Murray.

British curve associates a rise of 10 per cent above the pre-war adjusted price with an increase in output of some 13 per cent, and a fall of 10 per cent with a decrease of between 6 and 7 per cent.

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PARETO'S SOCIOLOGY

By Max Millikan

Those familiar with the brilliant achievements of Vilfredo Pareto in mathematical economics looked forward eagerly to the appearance of the English translation of his excursion into sociology, expecting to find the field treated at last with precision and clarity. To such readers, a first perusal of his four volume treatise on *The Mind and Society* was a disappointing experience. The author's admirers had for months been busily engaged in spreading the rumor that this was a Big Book, a book which would revolutionize not only sociology but psychology, government, science, philosophy, indeed, almost every conceivable branch of learning. Reviewers, most of them unable or unwilling to understand the work, concluded that it must be incredibly profound and arbitrarily adopted numerous phrases to label its significance. Pareto has been called in turn the Karl Marx of Fascism, the father of true social science, a profound student of scientific method and a penetrating analyst of human nature. With these superlatives ringing in his ears the reader plunges into what seems at first the most obscure and confused blend of sociological theory, classical antiquarianism, sheer vituperation and sage comment on everything from metaphysics to obscenity. The natural scientist has been led to expect a scholarly analysis of scientific method; he finds in the first chapter evidences of a somewhat naive empiricism. The sociologist looks for a clearly defined and consistent structure of theory explaining social forces in detail; he finds what seem to be merely rough generalizations restating practical maxims. The politician thinks he is getting a concise handbook of practical politics; he is obliged to paw over pages and pages of material irrelevant to his purpose in order to extract an occasional, parenthetical bit of advice to rulers.

Were there really little of importance in Pareto's work, this blasting of false hopes would not be serious. The unfortunate thing is that the initial disappointment for which the first volume is primarily responsible may very well blind us to the true magnitude of Pareto's contribution. If I appear to concern myself unduly with what Pareto and his concepts do *not* involve, it is in the hope that prospective readers may approach the book with a full realization of its inadequacies. Prepared thus for the arduous and often irritating task that lies before them, they may have the patience to dig through the welter of inconsistency and superfluity which envelopes the very meaty substance of the author's thesis.

That the book has been hailed as so many things gives some indi-

Vilfredo Pareto
1848—1923

cation of the heterogeneous nature of its contents. Pareto, dissatisfied with the abstraction and limited application of mathematical economics, wrote the *Trattato* in his later years in an attempt to find some more complete and adequate explanation of social forces. That much of it was left in the form of lecture notes explains in part its loose, not to say casual, structure. Further, it was written over a period of fifteen years and was never revised. As the author's ideas evolved, which they did quite rapidly, inconsistencies among the various sections of the work inevitably crept in. As reviewers have, alas, discovered, it is possible to find almost anything in Pareto in isolated passages. Nonetheless, the careful reader will find a strand of consistent theory running through the whole.

Having given this preliminary warning that the book must be approached with determination and tolerance, we are ready to extract and evaluate the substance of what Pareto has to say. He is addressing himself to the sociologist, the scientist and, in spite of his own denial, to the general reader, by turns. Let us examine his work from the point of view of each of these audiences.

I. PARETO AS SOCIOLOGIST

To avoid disappointment here we must again understand what Pareto was trying to do. To say that a man has written a Sociology tells no more about his aims than that he has written a book. What about society was he trying to find out? His problem, and this should be kept constantly in mind, was admittedly a restricted though very fundamental one. He was not trying to set forth a quantitative description of the institutions and customs of society as the statistical school attempts to do; he was not writing a Spenglerian theory of history, nor probing into the origins and evolution of society after the manner of Spencer. Above all, he makes no claim to be setting up anything like a complete explanation of the forces which determine the nature and operation of society in detail.

His particular problem was to study and analyze what he calls the non-logico-experimental conduct of men from the standpoint of its importance in society. His definitions of the term are inconsistent and confusing. If we take rational conduct to mean conduct based upon observation and logical analysis of the facts, we may consider "non-logico-experimental" to be roughly synonymous with "irrational." In his economic studies, Pareto had found necessary the assumption that human beings act rationally in pursuing their economic ends. He was lead into sociology by the realization that, although in the economic sphere this assumption is close enough to reality to be helpful, in most regions of human activity it is not even remotely true.

Most politicians and practical men realize in a vague way that what they would term superstition, prejudice, sentiment, and whim are far more important than reason and observation in determining what men believe and how they act. The sociologists have, in general, been less cognizant of the fact though most of them have recognized the existence of irrational actions. But, until Pareto's time, such actions had been generally considered a rather deplorable by-product of society, not one of the primary determinants of social change. It is this particular gap in our knowledge that Pareto set out to fill. He appears at times to be trying to plug up countless other holes, but it is on the basis of his success or failure in this fundamental endeavor that his significance as a sociologist must be judged.

His conception of the way to go about the problem was first to gather as many facts as possible about society. He then examines these facts to see what uniformities they exhibit, classifies the uniformities, and generalizes them into laws. Pareto calls this procedure the scientific method. We shall examine the validity of his method and its application to science in a moment, but first let us see what he does with it.

Since the fundamental aim of his study was the analysis of faulty human reasonings, he used as his factual source the verbal manifestations of reasonings, that is, the writings and sayings of people of the past. He was perhaps unduly influenced in this selection of evidence by the fact that he was a profound classical scholar and had this material at his finger-tips. Indeed there are times when the reader rather regrets the chronic insomnia which kept Pareto up, many times all night, poring over the works of Greek and Roman writers. The classical scholar occasionally obscures the sociologist.

After examining this wealth of material, Pareto tells us, he found certain uniformities, certain constant elements in all the reasonings about any given subject and certain variable elements based thereupon. For example, all discussions of the practice of baptism have a certain element in common, namely the belief that the integrity of the individual can be assured or restored by acts of some kind. The rites vary. In some places and times blood is used, in others, water. By some peoples, complete immersion is believed to be necessary; others let it go at a nominal sprinkling. But there is involved in all these rites a constant element which may be said to lie at the root of all the practices of baptism. Such constant elements Pareto finds throughout the reasonings of men. He calls them residues. He makes it plain that no significance is to be attached to the choice of the word "residues" to designate these constant elements. It is a label and nothing more. A letter or a number would serve equally well.

These residues are of various orders of generality. There are classes

and sub-classes, groups and sub-groups. Pareto was an ardent, if not always wholly apt, classifier. All the residues may be classed under six main heads of which the first two are the most important. I give them all in some detail in order to clarify their meaning.

The first is the Residue of Combinations. This is the impulse or propensity people exhibit to invent things, to associate things in new and different ways, to fuss with a number of elements trying to combine them into all imaginable forms. Inventors and scientists, speculators and enterprising business men, most crooks, and most philosophers possess this residue in very large degree. It is the residue which is responsible for the growth of most magic, the mysterious linking of certain objects and acts with certain alleged results, the use of omens and auguries. One of the most important residues of combination is that of the need for logical developments, the propensity human beings have to gloss over all their actions and beliefs with a coating of logic. We shall meet with this again later. The residue of combinations is the residue most prominent in enterprising and active peoples whose society rapidly changes its form.

Next comes the residue of the persistence of aggregates or, as Mr. Livingstone calls it, the residue of group persistences. It is this residue which lies behind all loyalties and all traditions connected with people, places, and institutions. The family, the state, and the nation are held together by the sentiments manifested by this residue. It is the symbol of all the preserving forces in a society, the forces which resist change and flux. It is responsible for the permanence of institutions and beliefs and the handing down of habits and customs from generation to generation. Conservatives, people who believe in the maintenance of the status quo, those who partake largely of the moral and ethical heritage of their predecessors all exhibit the residue of persistence of aggregates in considerable measure. It is the indication of the sentiment which impels men to keep things as they are.

The third category is the residue of the need of expressing sentiments by external acts. Mild expressions of this residue are all celebrations of great occasions, parades, jubilees, graduation ceremonies and the like. In its more extreme form it lies behind religious ecstasies and excesses of all kinds. It is the manifestation of the impulse to act for the sake of acting whenever one is undergoing deep emotion whether the action has any rational point or not.

The fourth and least satisfactory category Pareto calls the residue of sociality. It seems to be a sort of miscellany comprehending everything not covered by the other five heads. It includes the impulse to conform to the prevalent norms of society and the impulse to force others into such conformity. It embraces social manifestations of the

sentiments of pity and cruelty and the repugnance to suffering as well as of the tendency of self-sacrifice. The indications of sentiments of superiority and inferiority in the various classes of society and the impulse behind asceticism fall under this head.

Fifth we have the integrity of the individual and his appurtenances. This leads to all kinds of propitiation and revenge, acts to preserve the honor and position of the individual in his own eyes, the eyes of his fellows, or those of a supernatural power.

And lastly and least important, we have the sex residue which gives rise to most expressions of opinion on matters of sex morality and to acts intended to preserve or destroy the moral standards of a community.

I have described the residues at such length because I hoped that thereby I might convey a clearer notion of their nature. They are extremely difficult to grasp without misconception, and Pareto is not very helpful in defining them for us. One thing, I trust, has emerged from the above discussion. The residues are not, in the common or psychological sense of the word, instincts or sentiments. Nor are they the fundamental drives which motivate individuals. They are rather the manifestations of these things in social behavior. Pareto is not at all concerned with individual psychology. He would be the first to say that an adequate science of psychology would be a great help to sociology, but he would make no pretence that he was trying to solve that problem. He is examining human reasonings in their bearing on social organization as manifested in the verbal expressions of men and women. In the great mass of reasonings at which he looks, he finds certain elements which are constant. In other words, he finds that human reasonings may be explained by the hypothesis that people tend to reason and to act in social groups in certain ways. Those tendencies or group propensities are the residues. They differ from instincts in that there is no implication that they actually exist as drives or biological forces in society. The sex instinct, for example, is that very important and fundamental instinct which, according to biologists, impels people to the sex act. The sex residue, on the other hand, the least important of the residues, is merely the manifestation in human reasonings of a tendency to hold certain views on matters of sex—especially as they relate to the affairs of a group—and to act on those views. Once the general concept of a residue is made clear, the remainder of Pareto's analysis follows simply and logically. The nature of each of the residues and their many sub-classes is extensively explained with copious examples and illustrations in Volume II.

We next proceed to a consideration of the variable portions of human reasonings and we find that these too can be classified. These

variable portions Pareto calls derivations. We recall the subclass of the residue of combinations which has to do with the need people feel to give reasons for their acts and beliefs. These reasons are the derivations. Their classification, which occupies the first half of Volume III, is less important, because it is more obvious, than that of residues. Nevertheless, the discussion of these derivatives is one of the most illuminating sections of the work. What we know as logical sophistries make up only a small proportion of the common derivations. Most of them are much less complex than a good sophistry. Among the most common are simple assertion of fact, usually wrong, and appeals to authority. I believe in immortality because it is true, or perhaps because it is good; or I believe in democracy because a god, or my father, or all right thinking people believe in it. Another group is the support of a belief because it is in accord with certain sentiments or principles such as Americanism or the good of the whole. And finally we have the great swarm of more complicated verbal proofs resting on indefinite terms, muddy concepts, emotive language and the subtler sophistries of reason. One of the best sections of the book is on the derivations involving the use of ambiguous metaphor, allegory and analogy.

These things are not new to us in and of themselves. Anyone who reads the Hearst papers or listens to a good political speech is conscious that derivations of all sorts are being used. But comprehensive analysis of such reasonings in terms of something more fundamental were rare indeed in Pareto's day. His discussion of derivations is all built upon the foundation of his analysis of residues and is thus given a coherence and significance which previous attempts at a similar analysis had not had.

Of considerable importance is the emphasis Pareto places on the fact that the derivations are secondary factors. It has been felt by many that they were primary, that is that if the logical inconsistency or inadequacy of a reasoning is sufficiently well demonstrated, the theory which depends on it will be abandoned. On the basis of Pareto's theory this is not necessarily, or even usually, the case. Since the residues are the fundamental elements underlying beliefs, you must alter the residue in some way in order to change the belief. The destruction of a particular derivation will merely lead to the establishment of another to prove the same point.

These then are the building blocks of Pareto's theory. They are a little more than tools of analysis, but they are by no means a science of sociology. It was not his aim in the Tratatto to erect such a science. The residues, he points out, constitute only one of the many factors affecting the state of a society at a given moment, albeit a very important one. Physical influences, the effects of race and abilities, con-

tacts with other societies, together with many other factors, most of them interactive, would have to be studied before we could arrive at a comprehensive picture of social forces. He gives, in the last chapter of Volume III and the first chapter of Volume IV, a very rough and descriptive discussion of the place of residues and derivations in social analysis and of the evolution of these qualities.

He attacks the problem of general social analysis by setting up the concept of the social equilibrium. This is merely the concept, familiar to economics and the natural sciences, of a state such that any disturbance sets up forces which tend to restore the original state. Pareto considers the kinds of fluctuations in the social equilibrium which changes in the residues will bring about, first from the standpoint of virtual movements, that is, movements which would take place if everything except the force under consideration remained constant, and second from the standpoint of real movements. The theoretical aspects of this discussion are very vague and confused and could hardly be said to be an application of the author's theory.

His discussion of the way residues and derivations change through time is largely historical rather than analytical. That is, he shows not how from their nature they ought to change, but rather how from historical studies we see that they have changed. Nonetheless, as an illustration of his general thesis, this discussion clearly deserves a place in the development of the theory. Beyond this in Volume IV—an addition of great interest to the general reader but of little importance to sociological theory—he uses his terminology to discuss a great many matters, historical, political and philosophical, under the guise of further illustration and elucidation. Most of this material has to do in one way or another with non-logical action, but it is not closely related to his theory and serves, on the whole, rather to confuse than to illuminate his concepts.

II. PARETO AS SCIENTIST

Pareto has been acclaimed, wildly and incautiously, as the first great social scientist. At last, so pant the reviewers in adoration, the method of the natural sciences, so fruitful in the physical realm, has been applied to the social sciences by a man who understands it from top to bottom. The reason for this acclaim is not far to seek. Pareto himself opens his work with a long chapter on scientific method in which he announces that he proposes to use it. All through the work at frequent intervals he calls his readers attention to the fact that he is using it, and finally at the end he concludes happily that he has used it. The trusting reader can scarcely hold out against this sort of persuasion.[1]

[1] Poincaré says "Nearly every sociological thesis proposes a new method

In attempting to make a dispassionate evaluation here, we must answer two questions: first, does Pareto contribute anything new to the explicit statement of the method in the abstract, and second, does he use it himself? Anyone familiar with the works of Whitehead, Russell, Cohen, or even such ancients as William Stanley Jevons and the elder Keynes is compelled to answer the first question in the negative. Pareto may have understood the method of the natural sciences; there are indications in his work that he did. But his statement of it is at the least seriously misleading.

We must distinguish here between what is loosely termed the scientific attitude and what we may designate precisely as the scientific method. The former is an attitude of mind; the latter is a fairly clearly delimited methodology. The scientific attitude includes most of the obvious characteristics of the good scientist: a completely objective mind, a willingness to abandon any theory in the face of one better able to explain the facts, a healthy scepticism toward any statement not adequately demonstrated, a stern insistence on rigorous definition but an understanding of the purely arbitrary character of definition, and so on. All of these things Pareto seems to be able to state with admirable force and clarity. His attacks on many previous writers on the ground that they failed to exhibit some or all of these characteristics are wholly justified. How a man so well aware of what the scientific attitude involves could desert it in practice so frequently as does Pareto is a little difficult to understand. His definitions are for the most part eminently vague and occasionally contradictory, his statements are in many detailed matters inconsistent with each other, and his own personal views come popping out of the pages of *The Mind and Society* with explosive violence. Possibly this is to be ascribed to the fact that he never chose to revise his original manuscript. This continual emergence of the author's personality is the more unfortunate from a scientific standpoint because the substance of his theory, abstracted from its statement, is really quite free from personal bias and prejudice.

When we come to consider Pareto's treatment of the Scientific Method, we have a sadder story to tell. So much attention has been given to the subject that we are justified in considering a contribution significant only if it is very highly refined and its implications precise. Pareto's exposition not only lacks both these qualities but is in addition very seriously misleading.

Neglecting subtleties for the moment we may distinguish historically two points of view concerning the nature of the process of scientific in-

which, however, its author is very careful not to apply, so that sociology is the science with the greatest number of methods and the least results." *Science and Method*, ch. I, p. 19.

vestigation. The more primitive of these, in revolt against the speculative orgies of the middle ages, exalts induction as the sole path to scientific truth. Science, as conceived by the proponents of this view, is at least in its early stages completely divorced from hypothetical assumption. The scientist proceeds by examining systematically as vast a collection of facts as he can gather together. When he has all these facts before him he will perceive without speculation certain uniformities. He classifies them on the basis of these uniformities, generalizing the classification into laws. This is the method of science as put forward by Francis Bacon. R. D. Carmichael[2] says of it, "It has been pointed out with perfect justice that science in its progress has not followed the Baconian method, that no one discovery can be pointed to which can be definitely ascribed to the use of his rules." For "his mind was still enslaved by the formulae of quasi-mechanical scholastic logic. He supposed that natural laws would disclose themselves by the accumulation and due arrangement of instances without any need for original speculation on the part of the investigator."[3] While Pareto in one or two instances specifically rejects this point of view, the whole structure of his work as well as the better part of his explicit, methodological discussion gives the reader a conception of scientific method much akin to that of the early inductivists.

"Our first effort," he says, "will be to classify [facts] for the purpose of attaining the one and only objective we have in view: the discovery, namely, of uniformities (laws) in the relations between them. When we have so classified kindred facts, a certain number of uniformities will come to the surface by induction; and after going a good distance along that primarily inductive path, we shall turn to another where more ample room will be found for deduction. So we shall verify the uniformities to which induction has carried us, give them a less empirical, more theoretical form, and see just what their implications are, just what picture they give of society."[4] And again, "We are following the inductive method. We have no preconceptions, no *a priori* notions. We find certain facts before us. We describe them, classify them, determine their character, ever on the watch for some uniformity (law) in the relationships between them."[5] Once more, "We start with facts to work out theories, and we try at all times to stray from the facts as little as possible. We are looking for the uniformities presented by the facts, and those uniformities we may even call laws."[6]

[2] *The Logic of Discovery*, p. 7.
[3] *Encyclopaedia Brittanica*, 11th Ed., Vol. 14, p. 502.
[4] Sect. 144, p. 72.
[5] Sect. 145, p. 75.
[6] Sect. 69, p. 34.

That Pareto should have adopted this point of view is the more surprising since many of his contemporaries had recognized its weaknesses. As early as 1874 we find Jevons saying, "Within the last century a reaction has been setting in against the purely empirical procedure of Francis Bacon, and physicists have learnt to advocate the use of hypotheses. I take the extreme view of holding that Francis Bacon, although he correctly insisted upon constant reference to experience, had no correct notion as to the logical method by which from particular facts we educe the laws of nature. I endeavor to show that hypothetical anticipation of nature is an essential part of inductive inquiry and that it is the Newtonian method of deductive reasoning, combined with elaborate experimental verification, which has lead to all the great triumphs of scientific research."[7]

Briefly, the error, both of Bacon and of Pareto, was in believing that induction could be divorced from deduction. Bacon went so far as to banish deduction entirely; Pareto relegates it to the later stages of scientific inquiry. Actually both forms of reasoning—if, even ideally, they can be differentiated—are inextricably intertwined from the very start of investigation on down. Bacon believed that it was possible to *discover* a unique classification in the data of observation, that a particular set of uniformities were in some sense inherent in phenomena. All that was necessary to set up scientific systems was to look at the facts and record the uniformities found therein.

He overlooked the fact that the very search for a uniformity involves the presence in the mind of the investigator of a concept in terms of which the facts are to be judged uniform. No two observations are ever exactly alike. When we say that a set of facts possesses a uniformity, we mean that there is a certain idealized attribute which each of them possesses in approximation. The process of classification consists in first choosing a set of such attributes as criteria and then sorting the facts according to whether they do or do not possess these attributes. Clearly this choice is initially somewhat arbitrary. The observer is confronted with numerous alternative possibilities of classification. Thus in a sense he does not *discover* the unique set of uniformities immanent in the facts, but rather *imposes* uniformity on the facts by the adoption of certain ideal rules of selection. Pareto unconsciously recognizes such selection when he says, "We keep open house to all facts whatever their character," and then renders the statement absurd by adding, "provided that directly or indirectly they point the way to a uniformity."[8]

[7] William Stanley Jevons, *The Principles of Science*, Preface to the 1st Ed., p. 8.

[8] Sect. 81, p. 44.

The success of a science will depend on the choice of these rules. Certain classifications will be found to be more useful than others. As yet there has been discovered no logical method for determining beforehand what hypotheses and assumptions will lead to fruitful analysis. The process is to a considerable extent one of trial and error. The scientist looks about him at a welter of undifferentiated fact. By some intuitive process, not at all understood, assumptions suggest themselves to him on the basis of which order can be brought out of chaos. A very little investigation will serve to indicate that certain of these assumptions will not be useful. These are discarded and others pursued further. The conceptual background of the investigator will determine to some extent what assumptions he initially makes. Scientific genius consists largely in the ability to make fruitful initial assumptions, divorcing oneself as much as possible from the traditional conceptual background.[9] Pareto makes frequent reference to the value of hypothesis, but he seems to view it rather as a useful tool which may or may not be used than as an element which is consciously or unconsciously present in all investigation.

This weakness in Pareto's thought is of more than philosophical importance. The adoption of his method serves, by obscuring a portion of the structure of theory, to slow up theoretical advance and may even lead to serious blunders. For science progresses in large part by critical examination of existing theory. When a principle is thrown into question by an experiment, there are two places to look for error. Either the logic by which the principle has been derived is faulty, or the assumptions on which the logic rests are inadequate. It is thus necessary for the sake of clarity that the full set of assumptions on which a theory rests be clearly and explicitly stated as assumptions. But the pure inductionist does not admit that his initial classification and selection of uniformities rests on any assumptions whatever. It

[9] Whitehead states very concisely the difficulty with the Baconian view. "Our coordinated knowledge, which in the general sense of the term is Science, is formed by the meeting of two orders of experience. One order is constituted by the direct, immediate discriminations of particular observations. The other is constituted by our general way of conceiving the Universe. They will be called the Observational Order and the Conceptual Order. The first point to remember is that the observational order is invariably interpreted in terms of the concepts supplied by the conceptual order. We inherit an observational order, namely types of things which we do in fact discriminate; and we inherit a conceptual order, namely a rough system of ideas in terms of which we do in fact interpret. We can point to no epoch in human history, or even in animal history, at which this interplay began. Also it is true that novel observations modify the conceptual order. But equally, novel concepts suggest novel possibilities of observational discrimination."

A. N. Whitehead, *Adventures of Ideas*, ch. IX, p. 198.

comes, he maintains, directly out of the facts. Hence, in his exposition the assumptions actually there are obscured and hidden away beneath a screen of apparently untreated observational fact.

Thus Pareto's own theory contains what appears to be an actual flaw arising from his misunderstanding of how he arrived at it. He thought that in the residues he was *finding* what is in some absolute sense a constant element in human reasoning. He spends some time attempting to demonstrate with factual proof that the residues do not appreciably vary while the derivations do. Actually the process of finding the residues was one of postulating the residues. In one chapter he undertakes a long analysis of the rites and practices followed by the Greeks in the hope of controlling storms. He emerges at the end of this study with the conclusion that in all these rites and practices there is a constant element, namely, the belief that in some way storms can be controlled by divers rites and practices. But this of course is what he started out with. His conclusion is thus tautological. In effect he sets out to look for a constant element. He looks at all the facts that contain element *a*. After an exhaustive study he concludes that all these facts contain a constant element *a* while the other elements vary from fact to fact. Therefore *a* is constant.

The difficulty lies in the fact that he appeared to forget that classification is initially an arbitrary process, that you can adopt any criteria you wish by which to separate your facts into groups. Clearly once you have a classification, all the facts in one group will contain the element by which they were selected. It is thus possible to say that for the purpose of that classification, that element is constant. But it would be well to realize that you might have arranged your facts differently and got different constant elements. For instance, Pareto might have used as the criterion for a classification one of his derivations, putting into one category all those human reasonings in which an appeal is made to some sort of authority. Examining this class of facts, he would then have found a number of his residues. It would then have seemed to him that the constant element is the appeal to authority and the variable element the residues. Of course, pointing out that the constancy of the residues is tautological in no way affects their usefulness. One of the most useful concepts in physics, that of the conservation of energy, is regarded by many physicists as essentially a tautological concept.

Less fundamental but equally illustrative of Pareto's misunderstanding of scientific method is the form of his exposition which tends to obscure rather than to bring out the significance of his theory. Whatever excuse there may be for using the Baconian method in investigation, there is none whatsoever for using it in presentation.

The first volume of *The Mind and Society* presents great masses of facts with the expectation that the reader, without any assistance, will find the author's uniformities and adopt his criteria for classification. In selecting and setting down his facts, Pareto had in mind at least roughly the conceptual framework into which he intended to fit them. He withholds this framework from the reader until Volume II, believing that the reader will in some way induce it from the facts. There is no foundation for this hope. As already pointed out, it is the genius of the scientist that observations suggest to him fruitful assumptions. To expect the same observations not only to suggest assumptions to the great body of readers but also to suggest identical assumptions is indeed inductivism gone astray. The attempt can give rise only to the kind of confusion which makes Volume I difficult. Pareto should have overcome his almost pathological fear of scholasticism and stated his theory in its abstract form, clearly and without hedging, at the very beginning so that we might know, throughout the rest of the book, what he was talking about.

In one more respect Pareto's misunderstanding of scientific method led to what is perhaps a clumsy statement of the theory. He felt that the facts he was adducing were to be considered as evidence rather than as illustration. That is, he felt that his task was to prove his theory by reference to the facts, not simply to illustrate the nature of his concepts by examples. Now actually you cannot prove a theory by the facts. You can only disprove it by the facts. It is always possible to select great masses of facts to support a theory however tenuous. Witness Spengler. Hence such an amassing of facts, while in this case entertaining, is scientifically extraneous. It is the practice in the natural sciences for investigators to adduce a certain amount of evidence for their discoveries. This is useful only because most workers in the field can be trusted to be reasonably good critics of their own work. In sociology, where this is patently not the case, a number of disinterested persons must confess themselves unable to refute a theory before its probable value can be established. If Pareto had realized this, he might have greatly reduced the number of historical examples he uses, employing only those which have high illustrative value; and he might have substituted for some of them hypothetical cases designed merely to clarify his meaning.

We may conclude then that Pareto understood the scientific *attitude*, though he did not always adopt it; but that his exposition of scientific *method* is so faulty that it should be kept from tender young minds.

III. PARETO AS ESSAYIST

Here we can return to the role of advocate which we have been obliged unpleasantly to desert for a short time. In small doses *The Mind and Society* is a stimulating, intriguing and often amusing book to read. The characteristic which makes it bad science, the constant emergence of the author's pungent personality, makes it excellent reading. There is stinging irony and occasionally fine invective splattered through its pages. The antiquarian and the classical scholar will delight in the wealth of learning which pours forth from every chapter; even the layman not versed in these things will find entertaining the many bits of curious information and the countless anecdotes of which the author is so little sparing.

On numerous subjects really outside the purview of his treatise, Pareto presents in the terminology of residues and derivations points of view and scraps of advice which have given rise to the statement that no member of the ruling classes can afford to be without a copy of the book. To go into any detail in any of these matters would be to quote large sections of the work; they include such subjects as the nature of political corruption, the forms of government, protection and free trade, democracy and the use of force in society. His position on this latter point, expounded in perhaps fifteen of the book's two thousand thirty-three pages, has given him the title, the Karl Marx of Fascism.

IV. CONCLUSION

That the sociology of Pareto is not, as is so often averred, primarily significant for its contribution either to the statement or the application of scientific method should be clear. That he has written a fascinating and engaging set of volumes including a vast amount of penetrating comment and sage advice for leisurely perusal is also certain.

It is unfortunate that he made no attempt to integrate his work with that of previous sociologists. Until his theory is worked into the fabric of the field, it is impossible to measure his significance with any accuracy. Certainly we can say, however, that he has contributed to the precise statement of the nature, effects, and importance of irrational action and, by his emphasis on this phase of social activity, he has corrected an unfortunate lack of balance in the field.

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DEMAND FOR BOOTS AND SHOES AS AFFECTED BY PRICE LEVELS AND NATIONAL INCOME

By Victor S. von Szeliski and L. J. Paradiso

Object and conclusion of the study. The present study attempts to develop demand curves for shoes and find the factors on which the total income of the industry depends. The main conclusion of the study is that national income is of overwhelming importance in determining the income of the shoe industry and that prices are of less importance. If national income increases by 10 per cent, other factors being equal, the income of the shoe industry will increase by 7.4 per cent; if prices increase by 10 per cent, other factors remaining the same, the income of the shoe industry will then increase by 4.8 per cent.

The Data. The analysis covers the years 1919 and 1921–1933, data for 1920 not being obtainable. Table 1, column I, shows the number of pairs of leather shoes produced for the domestic market, i.e., production adjusted for exports and imports. The types of shoes included are men's, women's, youth's, misses', infant's and athletic. Slippers are excluded. The figures for 1919, 1927, 1929, 1931, and 1933, are from the Census of Manufactures and are, if not accurate, at any rate the best obtainable. The intercensal year figures for 1926 through 1932 are nearly as good, as they are based on the Bureau of Census compilation (monthly series) tied in to the Census of Manufactures, as shown in Table 4 in appendix A. The numbers of pairs shown for 1921 to 1925 are less satisfactory because there are no Census figures to tie into. However, it is believed that the figures are accurate to within a few per cent.

Table 1, column 2, shows the total value of leather boots and shoes produced for the domestic market. For Census years this was derived by adding the value for the classes of shoes in question and correcting for exports less imports. For intercensal years the process involves an interpolation which is explained in Appendix A. Column 6 shows the value per capita.

Column 7 of Table 1 gives an index of national income, weighted by types of income. Since different types of national income are not equally available or are not spent with the same velocity for consumer's goods, an index of national income (explained in Appendix B) was especially constructed for studying the demand for consumer's goods.

Finally, column 8 of Table 1 gives the Bureau of Labor Statistics index of wholesale prices of boots and shoes. This index is quite unsatisfactory. The quotations are prices at the factory and are not for representative shoes. Nevertheless, it is the only price series available

which covers a long period. Another serious difficulty is that this is a wholesale price series, and not a retail. It is the retail price—the price paid by the consumer—which determines whether he buys or not, and how much.

TABLE 1
PER CAPITA DOMESTIC PRODUCTION OF LEATHER BOOTS AND SHOES

	(1) No. of Pairs Produced (Millions)[a]	(2) Total Value (Millions of Dollars)[b]	(3) Value per Pair (Dollars)[c]	(4) Mid-Year Population of U. S.[d] (Millions)	(5) No. of Pairs per Capita	(6) Value per Capita (Dollars)	(7) Index of National Income 1929 = 100	(8) Index of Prices BLS 1926 = 100
1918	257			104.4	2.40R		84.1	97.9
1919	270	998	3.70	105.0	2.57	9.51	91.3	134.7
1920	275			106.0	2.60R		99.8	151.1
1921	250	770	3.08	108.5	2.30	7.08	75.7	111.5
1922	297	791	2.66	110.0	2.70	7.18	83.0	98.1
1923	314	901	2.87	111.7	2.81	8.06	94.4	99.1
1924	282	837	2.97	114.0	2.47	7.34	95.2	98.4
1925	289	837	2.90	115.6	2.50	7.25	99.7	100.5
1926	294	825	2.81	117.2	2.51	7.05	99.9	100.0
1927	315	861	2.73	118.7	2.65	7.23	97.3	102.6
1928	312	864	2.77	120.2	2.60	7.20	97.9	109.9
1929	331	908	2.74	121.6	2.72	7.45	100.0	106.3
1930	271	728	2.69	123.1	2.20	5.92	89.6	102.0
1931	279	610	2.19	124.1	2.25	4.93	72.8	93.7
1932	268	466	1.74	124.8	2.15	3.74	56.1	86.3
1933	294	492	1.67	125.7	2.34	3.91	55.4	90.2

[a] For Census years 1919, 1927, 1929, 1931, and 1933, these are production as given in the Census of Manufactures corrected for imports and exports. For non-Census years the Bureau of Census production figures are used adjusted to Census of Manufactures and for imports and exports. The details of these adjustments are shown in Table 3. For 1920 the number is roughly estimated from leather production.

[b] For Census years from the Census of Manufactures adjusted for value of imports and exports. For non-Census years value is obtained from estimates of value per pair given in column (c).

[c] For Census years this is value divided by number of pairs. For non-Census years value per pair is estimated from "Gross Income" of Boots and Shoes Companies, and value of exports per pair as shown in later tables.

[d] *Statistical Abstract of the United States.*

R Rough estimate on slender evidence.

Types of leather shoes included are men's, women's, youth's, misses', infant's and athletic.

General Relations Among The Variables. Figure 1 shows the course of production per capita, value of production per capita, value per pair, prices, and the national income index. Per capita consumption is fairly stable over the period, the extreme variations falling within a range of about 13 per cent above and below the average of 2.48 pairs per capita per annum. Apparently there was a slight downward trend in the consumption curve; the 1929 peak is lower than the 1923 peak. It may be that the increased use of automobiles and decline in walking has on the whole lessened the demand for leather shoes, for, although women are using more pairs per annum, men are using fewer. The shift in any one year is small, but it is perceptible over the period.

Consumption data are non-existent, and we are thrown back on production as an approximation to consumption. This is justified provided that year-to-year changes in stocks are small. Are they? There is no way of telling, but with two definite seasons per annum and 2.5 pairs purchased per capita per annum, it would seem unlikely that the production and consumption would get seriously out of balance for more

than six months or one season, at a time. Hence, in two successive seasons, or one calendar year, there should be a close approach to balance of production and consumption.

No close relationship between per capita consumption (i.e., production), and price and income is observable. Broadly speaking, the main waves of shoe production do correspond to the main waves of the national income, but there is no close correspondence such as one ex-

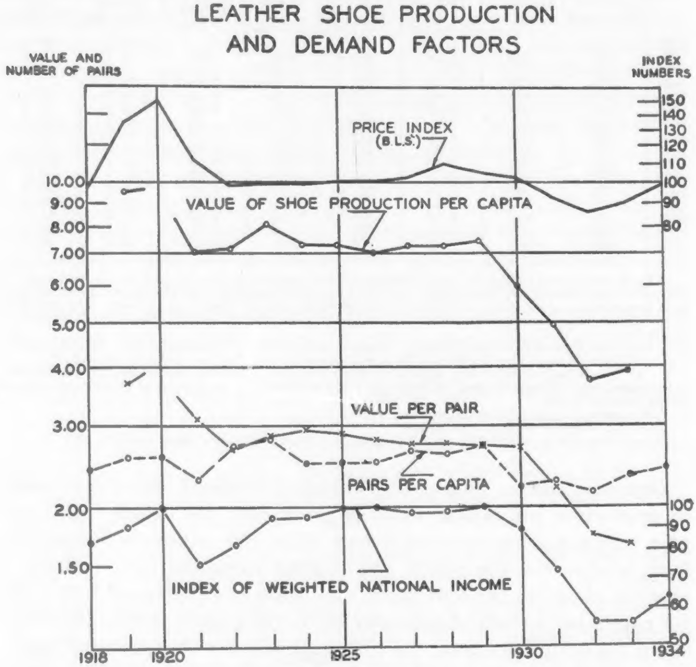

FIGURE 1

pects or desires in order to establish a mathematical demand curve. Price has a very slight relationship of the inverse type required by theory. For instance, the high price of 1928 is reflected by low production in that year.

This lack of a close inverse relationship between price and consumption on a per pair basis is what one might expect in a consumers' good available in several price brackets. This range of prices is so wide that the consumer with a decreased income can buy as many pairs of shoes

as before if he is willing to go into the next lower price bracket, or buy fewer shoes of the same or higher quality.[1]

Thus, prospects do not appear bright for identifying the customer's choice as regards number of pairs purchased, but the situation is more encouraging for value per capita. The value line on the chart is clearly related to prices and income. It appears that the consumers tended to set aside a certain fraction of the national income for the purchase of shoes, depending on the price of shoes, and that considerable doubt exists only as to the *way* in which this fixed or determined fund is spent, whether for a few pairs of shoes at a high price, or more pairs of shoes at a low price.

A study of these curves also suggests that the consumer can and does build up an inventory of unused shoe leather during a certain period, and then slows up his purchases in the following year, and vice versa. There should be, therefore, some tendency for consumption to be inversely correlated with consumption in adjacent years.

The demand curve is usually written with quantity as the dependent variable. But in this instance it will be better to select value (quantity times price) as the dependent variable, inasmuch as the preliminary analysis suggests that value rather than quantity is what is determined. The equation chosen was

(1) $v = A\ T\ p^{\alpha}\ i^{\beta}$,

where v = annual value per capita,

p = BLS price index for boots and shoes,

i = weighted national income per capita,

T = a time trend, and

A, α, β are constants.

There are two general criteria which point to this as a suitable equation.[2]

[1] Thus the problem of the analyst is not to predict a number, but a frequency distribution according to price classes.

[2] These considerations do not prove that this is the unique, one and only, equation that can be used, but merely that it is superior to most others. Such theorizing to select an equation may appear fine-spun. But it is well worth while because

(i) It is just by such considerations that we get away from empirical equations and get "semi-rational" equations.

(ii) By tying down the curve by these extra-observational considerations we help to get it into the "true" position. For instance, the requirement that the curve shall pass through (0, 0) is, in effect, an additional and exact observation.

(iii) Extrapolation beyond the range of the observations is much safer—in fact, it is not possible at all with an equation that is merely empirical.

(iv) Even if the location of the curve is not much improved, its slope, particularly at the ends, may be very much improved.

(a) *The equation must be plausible for extreme values of p and i.* For instance, if the industry gives its products away for nothing, p is 0 and, hence, v must be 0. Similarly, the value per capita must be 0 when $i=0$.

(b) *The "homogeneity" condition—If the price and purchasing power factors[3] are all increased in the same proportion, the dependent variable must also increase in the same proportion.* The applicability of this principle may be seen from these considerations: Suppose prices, p, and national income, i, are both doubled, and the factors related thereto: wage rates, bank deposits, currency in circulation, the stated amounts in contracts, and so on. The quantity of shoes purchased should not be changed, and so, with price doubled, per capita value must be doubled. This is true for equation (1) but only if $\alpha+\beta=1$.

However, it is unreasonable to demand that $\alpha+\beta=1$ in this instance, because (I) p and i are only estimates, and approximate ones at that, of prices and income, and we have to allow a little extra freedom in the determination of the constants on that account; (II) moreover, we have not taken account of all causal factors in which the monetary unit is involved, the price level of other consumer's goods, for instance. The requirement that the sum of the exponents shall equal 1 would apply only if all price and purchasing power factors were taken into account. For these reasons, α and β are left to be determined by the observations alone without any restrictions.[4]

The formula giving value per capita as a function of time, price, and income, is found to be[5]

[3] Price and income; "time" is not a cause, but a device for securing congruence of the general levels of the factors.

[4] Charles F. Roos, *Dynamic Economics*, pp. 228–230, places no restriction on $\alpha+\beta$, stating that for an equation $y=Api^{-\beta}$, where y is the quantity demanded p is price, and i national income, the exponent β varies from product to product, "increased purchasing power does not flow evenly to all products and wages."

Denoting the price level of other consumers' goods by g, we might be justified in demanding that $\alpha+\beta+\gamma=1$ in the equation

$$v=ATp^{\alpha}i^{\beta}g^{\gamma},$$

if we found it not significantly different from 1.

[5] Since the price appears implicitly in v on the left hand side of equation (2), one might object to the validity of the constant α obtained in p in this equation, on the ground that v and p are highly correlated and the intercorrelations between p, i, and t, might account for the exponent obtained. To show rigorously that equation (2) is not spurious the exponent of p was obtained by using the following form of equation (2):

$$\frac{v}{p}=AT\ p^{\alpha'}i^{\beta'}.$$

It is found that $\alpha_1'=-.38$ and $\beta_1'=.43$. It is obvious that the coefficient of p differs significantly from 0 and hence that v/p is correlated to p by an amount

$$v = .003261(1.0264)^{-t}p^{.4795}i^{.7371} \tag{2}$$

where t stands for time as measured from $1919 = 0$.

In Table 2 the actual and estimated values per capita as well as the residuals are given, and these are shown graphically in Figure 2. The average per cent deviation is 5.9. The number of degrees of freedom is 4 and the standard error of estimate in terms of absolute quantities is

TABLE 2

LEATHER BOOTS AND SHOES ACTUAL AND ESTIMATED VALUE OF PER CAPITA CONSUMPTION

	Value of Per Capita Consumption Actual	Calculated	Actual Minus Calculated	% Residuals		
1919	$9.51	$9.55[a]	$−.04	0.4		
1920					Average percentage deviation	=2.8%
1921	7.08	7.19	−.11	−1.6	Standard deviation	=$.234
1922	7.18	7.06	.12	1.7	Standard Error of Estimate	=$.278
1923	8.06	7.62	.44	5.5	Coefficient of multiple correlation	= .98
1924	7.34	7.45	−.11	−1.5	Reduction in unexplained variance	
1925	7.25	7.57	−.32	−4.4	when Price variable is added	.48
1926	7.05	7.38	−.33	−4.7	when Income variable is added	.92
1927	7.23	7.05	.18	2.5	when Time variable is added	.83
1928	7.20	7.21	−.01	−0.1		
1929	7.45	7.01	.44	5.9		
1930	5.92	6.19	−.27	−4.6		
1931	4.93	4.94	−.01	−0.2		
1932	3.74	3.84	−.10	−2.7		
1933	3.91	3.78	.13	3.3		
1934		4.32 E				

[a] Estimated from the regression formula: $\overline{V} = .003261\,(1.0264)^{-\text{time}}\,(\text{price})^{.4795}\,(\text{Income})^{.7371}$.

E The chances are about 2 out of 3 that the actual value for 1934 lies between $4.04 and $4.60 and that the total value of leather boots and shoes lies between 510 and 580 million dollars.

about 28¢ per pair. The chances are two out of three that the industry's 1934 income per capita was between $4.04 and $4.60. The calculated value is $4.32. It will not be possible to get the "actual" until the 1935 Census of Manufactures, and the 1934 and 1935 Statistics of Income are available. Considering that the original data may be off by 2 per cent and that the BLS index of shoe prices is not very good, the results on the whole are better than might be expected.

A determination of the number of pairs per capita was also made. This is given by

$$n = .05313(1.0071)^{-\text{time}}D^{.3470}i^{.3502}p^{-.1607},$$

where n is number of pairs per capita and D is $2.50 \div$ number of pairs per capita for the preceding year (representing stored-up-supply or lack of it). However, the coefficient of multiple correlation is low, indicating that other factors, such as the consumer's shift in his choice of quality, are also important.

that is greater than can be accounted for by the intercorrelations of p, i, and T. The expression v/p is not number of pairs, but number of pairs adjusted for quality. Compare p. 345 below. Note, also, that the exponent of i in this case does not differ significantly from that found in the formula f or number of pairs, n.

Partial Relationships: Value as affected by price and value as affected by National Income. If both prices and national income change, what will be their joint effect?

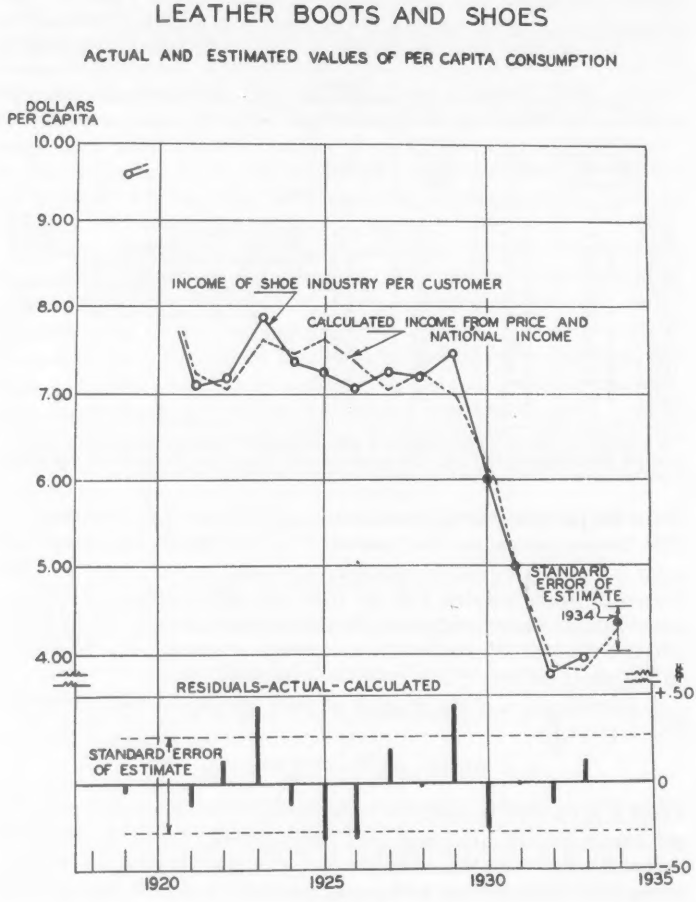

FIGURE 2

Boots and shoes are mainly bought by the national income produced by other industries, and the industry cannot influence this situation. But, even if the industry has to be a bystander in this regard, it may

yet be very much interested in knowing what is in store for it. The industry, as long as it is competitive, cannot do anything about prices either, but as it is to some extent collectively setting wages, and therefore costs, it does exercise an indirect control over the price. The decision as to the correct wage policy may thus depend on the answer to a demand curve problem of the type here being considered.

The effects of price, national income, and time, have been worked out by correlation analysis. A valuable technique for this is clearly explained in *Methods of Correlation Analysis* by Mordecai Ezekiel.

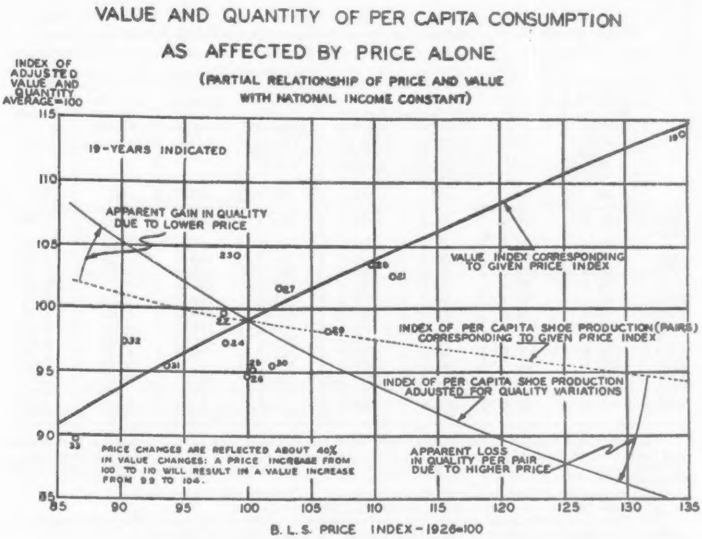

FIGURE 3

The method is entirely numerical in character, although charts have been used in this discussion to facilitate the presentation.

Figure 3 and Table 3 show the relationship between price and income based on equation 2, with the time element and the national income held constant and equal to their average values over the period. Apparently price changes are about 45 per cent effective in changing the total increase dollar intake of the Industry. The observations do not cluster closely about the heavy black line, indicating that price plays only an intermediate rôle in determining the industry's income.

If we divide both sides of the equation by price, we get quantity of

shoes purchased as a function of price and national income. This is the light solid line running from the upper left-hand corner of Figure 3 to the lower right-hand corner. It does not represent number of pairs purchased by the consumers but the "amount" of shoes purchased, the number of pairs adjusted for quality. We may look upon it as an index of shoe leather and shoe man-hours purchased by the comsuming public. The downward slope from left to right indicates that the public buys a lesser "amount" of shoes—less shoe leather and/or less man-hours of shoe labor, or less skilled labor, at higher prices, other things being equal.

TABLE 3

VALUES OF PER CAPITA CONSUMPTION ADJUSTED FOR AVERAGE PRICE, INCOME AND TIME

	Net Effect of Price				Net Effect of Income		Net Effect of Time	
	Index of Net Regression Values Average =100 (a)	Adjusted Values Average =100 (b)	Index of Per Capita Shoe Production adjusted by Quality Variaton (c)	Index of Net Regression No. of Pairs per Capita (d)	Index of Net Regression Values Average =100 (e)	Adjusted Values Average =100 (f)	Net Regression Values (g)	Adjusted Values (h)
1919	114.3	114.0	84.8	94.5	104.5	105.7	7.89	7.85
1920								
1921	104.1	102.6	93.4	97.3	91.1	91.0	7.48	7.36
1922	98.2	99.8	100.1	99.4	97.3	100.6	7.30	7.42
1923	98.6	104.3	99.5	99.4	107.2	115.2	7.11	7.52
1924	98.3	97.4	99.9	99.4	108.0	108.2	6.92	6.82
1925	99.2	95.1	98.7	99.1	111.5	108.7	6.75	6.46
1926	99.1	94.8	99.1	99.1	111.8	108.5	6.58	6.28
1927	99.1	101.8	96.6	99.1	109.5	114.1	6.41	6.57
1928	103.7	103.7	94.4	97.5	110.0	111.5	6.24	6.22
1929	101.8	108.3	95.8	97.3	111.8	98.0	6.08	6.46
1930	100.0	95.5	98.0	98.9	103.3	100.2	5.93	5.66
1931	96.2	95.5	102.7	100.0	88.5	89.3	5.77	5.73
1932	92.2	89.9	106.8	101.9	73.1	72.4	5.62	5.48
1933	94.2	97.4	104.4	100.9	72.3	76.0	5.48	5.66

Net Regression Values are calculated values from the equation with (a) price varying, and income and time held constant at their mean values; (e) income varying, and price and time held constant; (g) time varying, and price and income constant. (a) and (e) are then expressed as index numbers with average =100.

Adjusted values are net regression values increased or decreased by the residuals.

(c) These are the values given in (a) divided by the Price Index.

(d) These are based on the regression formula giving per capita consumption. The index is adjusted so that this line will pass through the intersection of the lines given in (a) and (c).

The actual number of pairs produced, as affected by price, is indicated by the dotted line. It has a very shallow slope, showing that price has but little effect on number of pairs bought. A 50 per cent increase in price results in a decline of only 7 per cent in the number of pairs produced. But if quality remained constant, the number of pairs follow the light solid line; a price increase of 50 per cent would then force a decrease of 20 per cent in number of pairs. Thus the sectors bounded by the dotted and light solid lines may be regarded as indicating quality variations.

There is a much closer relationship between national income and the industry's dollar intake. The observations cluster closely about the line

or relationship. Whereas an increase of 10 per cent in price only resulted in an increase of about 4 per cent in dollar intake, the same percentage increase in national income will increase it by a full 7 per cent.

VALUE OF PER CAPITA CONSUMPTION AS AFFECTED BY INCOME

(PARTIAL RELATIONSHIP OF INCOME AND VALUE WITH PRICE CONSTANT)

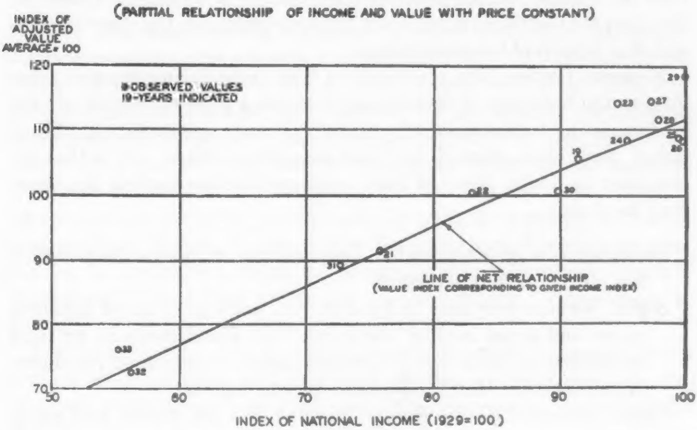

FIGURE 4

The following table summarizes the main statistical tests for importance of the constitutent factors.

RELATIVE IMPORTANCE OF INDIVIDUAL FACTORS AFFECTING BOOT AND SHOE VALUE PER CAPITA

Independent variables	Standard Error of Estimate	Coefficient of Multiple Correlation	Coefficient of Total Determination	Factors Already Considered	Factor Added	Coefficient of Partial Correlation	Reduction in Unexplained Variance
Time; Income	\$.40	.96	.93	Time; Income	Price	.69	.48
Price; Income	.65	.91	.83	Price; Income	Time	.91	.83
Time; Price	.97	.63	.40	Time; Price	Income	.96	.92
Time; Price; Income	.28	.98	.97				

Time and income, taken together, but omitting price, account for 93 per cent of the observed variance,[6] and price accounts for 48 per cent of the balance. The partial correlation coefficient of price and value per capita is, therefore, 0.69. Price and income taken together account for 83 per cent of the variance, and time accounts for 83 per cent of the

[6] *Methods of Correlation Analysis*, by Mordecai Ezekiel, pp. 178–185, may be consulted.

balance. Time and price together account for only 40 per cent of the variance, and income accounts for 92 per cent of the balance.

Since the values of the dependent variables for even years have been obtained by interpolation, the tests of significance could not logically be applied in this case, even were the further difficulty not present that the number of independent observations is unknown. However, it is clear that national income is a dominant factor in the shoe industry and that price is of less importance.

Neglected Factors. The presence of a time trend in any demand function, or the necessity of introducing it to get a good fit, should always be disquieting to the economic analyst.[7] It may mean that significant factors have been omitted, and that conclusions drawn are to that extent open to doubt. Some of these neglected factors causing the trend may be these:

Increased Use of Automobiles. Less walking; thinner soles possible; less workmanship necessary.

Styles. Women now have to have several pairs of shoes of different colors and styles in their wardrobe. This would cause an uptrend in number of pairs, but it has been offset by decreased purchases of men's shoes. Men's shoes have become lighter.

Other Consumers' Goods. Radios, gasoline, movies, liquor, and many other consumers' goods, have become increasingly attractive, and have successfully competed for a larger share of the consumer's dollars.

General Price Level. Lowered price levels for other consumers' goods would take the pressure off shoes and allow larger dollar expenditure for them, and vice versa.

Improved Manufacturing Methods. Insofar as these improved the quality of shoes without change in price, they would permit shifts to lower price brackets without sacrifice of quality. This would make for lower value per capita.

The net effect of these and the other neglected factors has been a declining trend of 2.64 per cent per annum.

Average Price per Pair Received by Manufacturers. It is of interest to the industry to know not only the income to be expected due to changes in wholesale prices of boots and shoes and in national income, but also the weighted average price per pair that may be expected because of such changes. This might be used to improve the scheduling of production as regards the relative proportion of various grades manufactured

[7] See Louis Bean, "The Use of 'Trends in Residuals' in Constructing Demand Curves," *Journal of the American Statistical Association*, March, 1932.

and the prices. A close determination of the average price per pair received by the industry in terms of the BLS wholesale price index of boots and shoes and national income is given by the following formula:

$$\text{Average price per pair} = .6626(1.209)^{-\text{time}} \times (BLS\ \text{price})^{.639} \times (\text{National Income})^{.49}.$$

If the BLS price index is increased by 10 per cent, other factors remaining constant, the average realized price may be expected to increase 6.4 per cent, the number of pairs decrease by 1.6 per cent, and the gross income of the industry increase 4.7 per cent. If the national income goes up 10 per cent, the average realized price goes up 4.9 per cent, even if quoted prices remain unchanged. When such an advance is in prospect, the industry can prepare for it in several ways, by shifting emphasis to the better grade shoes, by putting up prices, or various combinations of both. If cost curves for various grades of shoes were known, it would be possible to determine the production schedule which would be of most advantage to the industry.[8] Unfortunately, this interesting problem cannot be treated here.

The formula may also be used to make monthly estimates of the average price received, and these in turn may be multiplied into the reported production to estimate monthly gross dollar income of the industry.

Appendix A.

To Obtain Total Value of Leather Boots and Shoes for Intercensal Years

The process is essentially as follows:

(a) Reduce Census value of all boots and shoes to a per pair basis.
(b) Obtain regression equations of above with (i) export value per pair and (ii) gross income (from Statistics of Income) per pair, and so
(c) Estimate intercensal values, all boots and shoes, per pair basis, as described below.
(d) Compute Census value per pair of non-leather boots and shoes (obtainable for 1919, 1927, 1931, and 1933).
(e) Obtain regression equation of this with BLS index of all commodity prices and use the relationship to
(f) Obtain estimate of price per pair of non-leather boots and shoes for non-Census years.
(g) Multiply (f) by number of pairs of non-leather boots and shoes.
(h) Multiply (c) by number of pairs of boots and shoes.
(i) Subtract: (h) − (g) = Value of leather boots and shoes.
(j) Adjust (i) for exports and imports.

[8] Provided that national income can be forecast. A paper is in preparation dealing with this subject.

(k) Adjust number of leather boots and shoes for exports and imports. (k) is then put on a per capita basis (Table 1, column 5).

This process looks roundabout, but for accuracy it is better than it looks.

The method of interpolating intercensal values consists primarily of a combination of straight-line least-square fitting and the use of graphs. The method is quite general and is applicable to many other types of problems.

A "comparison series" is selected which is closely related to the "estimatand," y. (1) A scatter diagram is plotted and a regression line drawn for y as a function of x. The y estimated from x is denoted y'. (2) The residuals $r=y-y'$ or $r=y/y'$ (choice between the two depends on the conditions of the problem) are then plotted against time. (3) Interpolated values of r are then obtained for the desired times, graphically or by computation. (4) y'', the improved estimate of y, is then obtained by adding the residual to the calculated value of y:

$$y''=y'(x)+r.$$

The following comparison series were used:

(1) Gross income of boot and shoe companies from Statistics of Income, U. S. Treasury Department ÷ total pairs produced.

(2) Export value per pair.

(3) BLS index of all commodity wholesale prices.

TABLE 4

BUREAU OF CENSUS PRODUCTION ADJUSTED TO CENSUS OF MANUFACTURES AND FOR IMPORTS AND EXPORTS

	Total Boots and Shoes					Leather Boots and Shoes		
	Bureau of Census No. of Pairs (millions) (1)	Bureau of Census adj. to C. of M. No. of Pairs (millions) (2)	Exports No. of Pairs (millions) (3)	Imports No. of Pairs (millions) (4)	Supply for domestic market (millions) (5)	Bureau of Census No. of Pairs (millions) (6)	Bureau of Census adj. to C. of M. No. of Pairs (millions) (7)	Domestic No. of Pairs (millions) (8)
1919		331	22	0.2	309		292	270
1920			17	0.5				
1921	287	301	10	0.1	291	248	260	250
1922	324	340	7	0.6	334	289	303	297
1923	351	369	10	1.9	361	307	322	314
1924	313	329	9	2.7	323	274	288	282
1925	324	340	9	2.0	333	282	296	289
1926	325	341	8	2.4	335	286	300	294
1927	344	367	8	3.0	362	302	320	315
1928	344	361	7	4.4	358	300	315	312
1929	361	376*	7	8.4	377	315	330	331
1930	304	316	6	5.7	316	261	273	273
1931	316	316	4	5.9	318	265	277	279
1932	313	322	3	6.3	325	257	265	268
1933	350	350	3	4.3	351	293	293	294

Columns (7) and (2) Production for 1919, 1927, 1931, 1933, are Census of Manufactures figures. For non-Census years before 1927, Bureau of Census figures are raised by 5 per cent; for 1928 they are raised 5 per cent, 1930 by 4 per cent, 1932 by 3 per cent.

Columns (3) and (4) Bureau of Domestic and Foreign Commerce: *Commerce Yearbooks*, 1925, 1932.

Columns (8) Columns (3) and (4) applied to Column (7). Net exports of non-leather boots and shoes are very small and may be included without detriment.

* Includes estimate of 4½ million "other footwear" not included in 1929 schedule.

In Table 5, column (a), the gross income of boot and shoe companies is given, and Figure 5 displays a scatter diagram showing the census value per pair and the corresponding values per pair from gross income. The various points in the diagram are designated by their respective

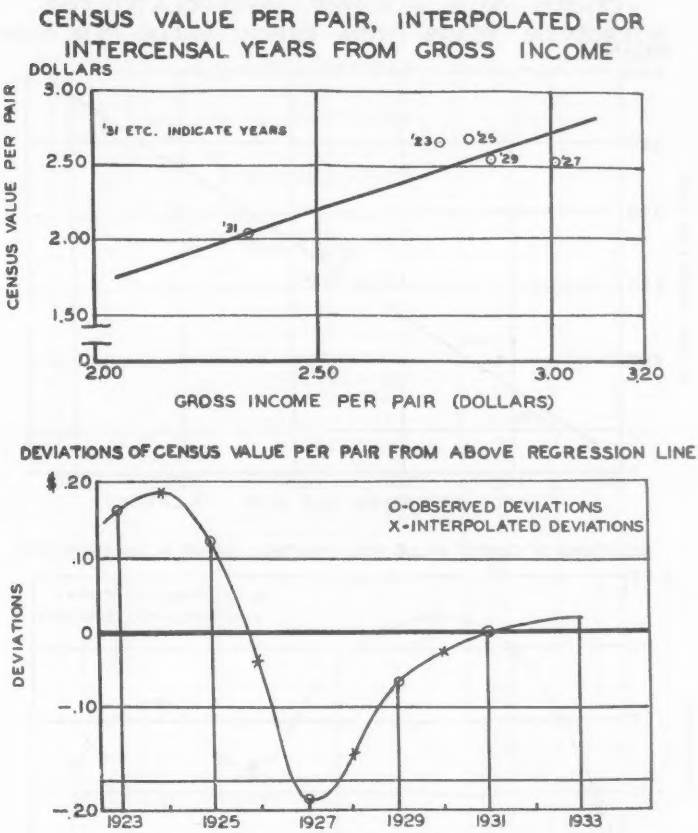

FIGURE 5

years. A graphical regression line is then drawn. The deviation of the census values per pair from the values given by the regression line is shown in the lower part of Figure 5. A smooth curve is drawn through the points found and its ordinates give the correction to the value determined from the regression line.

Estimates were similarly obtained by using export value per pair. Number of pairs exported are given in Table 4, and export value per pair are given in Table 5. Figure 6 shows the scatter diagram, the regression line, and the residuals.

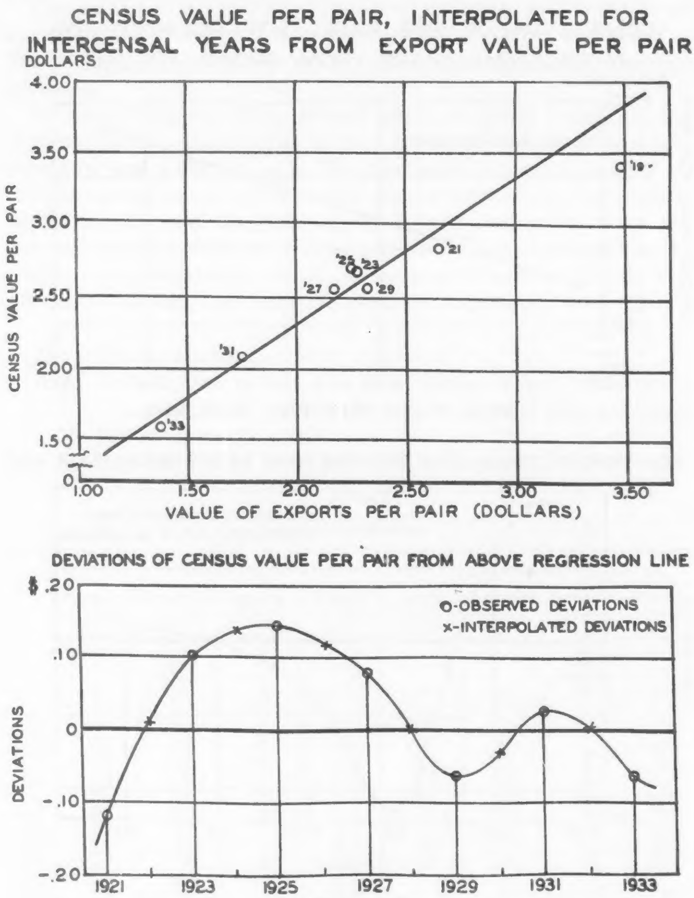

FIGURE 6

The value per pair of non-leather boots and shoes for the non-census years has been interpolated by the above method using the BLS index of all commodity prices as comparison series; this is given in Table 5, column (o).

TABLE 5
DATA USED TO ESTIMATE VALUE PER PAIR OF LEATHER BOOTS AND SHOES IN NON-CENSUS YEARS

Year	Adjusted Gross Income*	Census Value of Total Production*	Value of Exports*	Value of Imports*	Census Value per pr.	Gross Income per pr.	Value of Exports per pr.	Value per pr. estimated from Gross Income	Value per pr. estimated from from Exports	Average of (h) and (i)	Value of Total Production*	Value of Production for Domestic Market*	Footwear other than Leather Boots and Shoes: Number of Pairs (millions)	Footwear other than Leather Boots and Shoes: Census Value*	Footwear other than Leather Boots and Shoes: Value per pr.	Leather Shoes: Domestic Value*	Leather Shoes: Value per pair
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)	(k)	(l)	(m)	(n)	(o)	(p)	(q)
1919	$	$1,128	$75.4	$ 0.3	$3.41	$	$3.48	$	$3.41	$3.41	$1,128	$1,054	39	$56	$1.44	$998	$3.70
1920			67.7	1.0			3.97		4.02	4.02							
1921		850	27.5	0.6	2.82		2.65		2.82	2.82	849	822	41	52F	1.27E	770	3.08
1922	915		15.5	0.9		2.69	2.19		2.51	2.51	853	838	37	47F	1.26E	791	2.66
1923	1,020E	982	22.6	1.9	2.66	2.76	2.28	2.66	2.66	2.66	982	961	47	60F	1.28E	901	2.87
1924	960		19.1	2.8		2.92	2.24	2.85	2.65	2.75	905	889	41	52F	1.27E	837	2.97
1925	960E	910	20.1	2.9	2.68	2.82	2.26	2.68	2.68	2.68	911	894	44	57F	1.29E	837	2.90
1926	1,004		17.5	4.0		2.94	2.18	2.65	2.58	2.61	890	877	41	52F	1.28E	825	2.81
1927	1,104	932	18.0	5.9	2.54	3.01	2.18	2.54	2.54	2.54	932	920	47	59	1.26	861	2.73
1928	1,087		15.6	9.6		3.01	2.25	2.60	2.54	2.57	928	922	46	58F	1.27E	864	2.77
1929	1,079	959	16.1	18.8	2.55	2.87	2.33	2.55	2.55	2.55	959	962	46	54	1.17	908	2.74
1930	880		12.3	11.3		2.78	2.14	2.50	2.39	2.45	776	775	45	47F	1.05E	728	2.69
1931	740	647	7.5	7.0	2.05	2.34	1.76	2.05	2.05	2.05	648	647	39	37	.95	610	2.19
1932	586		4.5	2.8		1.82	1.45	1.52	1.72	1.62	522	520	57	54F	.95E	466	1.74
1933		548	4.2	2.4	1.57		1.38		1.57	1.57	550	548	57	56	.98	492	1.67

* Millions of Dollars.
(a) From Statistics of Income; E are adjusted values due to incompleteness in 1923 and 1925 and are estimated from gross sales of leather and leather products and the number of companies reporting.
(b) Census value of products less estimated non-shoe products.
(c) and (d) Bureau of Foreign and Domestic Commerce; value of imports is increased by duty paid.
(e) Column (b) ÷ column 2, Table 4.
(f) Column (a) ÷ column 2, Table 4.
(g) Column (c) ÷ column 3, Table 4.
(h) Value per pair indicated from Column (f) related to the Census value in odd years.
(i) Same, from average value per pair of exports.
(k) Column (j) × column 2, Table 4.
(l) Adjusted for value of exports less value of imports, column (c)–column (d).
(m) Column 5—column 8 of Table 4.
(o) Estimated from BLS all commodity price index tied into the Census figures in 1919, '27, '29, '31, and '33.
(n) Column (o) × column (m).
(p) Column (k)—column (n).
(q) Column (p) ÷ column 8, Table 4.

This method may be elaborated. The regression line need not be straight but it can be curvilinear if the conditions of the problem so indicate and the points so suggest. Instead of a direct interpolation a time trend can be fitted to the residuals, and the residuals from this time trend interpolated. Mechanical methods of interpolating the residuals may replace the graphical.

TABLE 6
NATIONAL INCOME INDEX COMPONENTS WEIGHTED BY RELATIVE AMOUNTS SPENT FOR ATTIRE BY RECIPIENT GROUPS
Data from King

	Wages ×weight 1 (millions) (1)	Pensions, etc. ×1 (millions) (2)	Salaries $\times\frac{1}{2}$ (millions) (3)	Entrepreneurial ×1 (millions) (4)	Property Income ×1/10 (millions) (5)	Weighted National Income (millions) (6)	Index of Weighted National Income per Capita 1929 = 100 (7)
1918	$20,414	$ 678	$2,808	$15,560	$ 908	$40,368	84.1
1919	23,029	794	2,894	16,302	1,020	44,039	91.3
1920	29,540	1,016	2,932	15,335	1,120	48,823	99.8
1921	22,500*	1,006	2,964	11,270	1,121	37,740	75.7
1922	24,553	1,097	3,012	12,093	1,185	41,940	83.0
1923	28,691	1,046	3,289	14,185	1,271	48,482	94.4
1924	29,051	1,243	3,550	14,745	1,287	49,876	95.2
1925	30,762	1,085	3,752	15,960	1,434	52,993	99.7
1926	32,256	1,173	3,954	14,857	1,514	53,754	99.9
1927	31,000*	1,229	4,170	15,164	1,549	53,112	97.3
1928	31,900	1,065	4,410	15,078	1,615	54,079	97.9
1929	34,485	1,000	4,441	14,188	1,690	55,904	100.0

* Lower than given figures because King's estimates in 1921 and 1927 are questioned.
Data from Dept. of Commerce Series. Source: Survey of Current Business, January, 1935.

SPECIFIED INDUSTRIES

	Salaries $\times\frac{1}{2}$ (millions)	Wages ×1 (millions)	Salaries Wages 7/10 (millions)	Compensations etc. ×1 (millions)	Dividends Interest ×1/10 (millions)	Individual Enterprisers ×1 (millions)	National Income Weighted (millions)	Index of Weighted Nation Income per Capita 1929 = 100
1929	$1,200	$15,000	$22,400	$ 900	$1,580	$13,800	$54,880	100.0
1930	1,200	12,400	21,490	1,000	1,510	12,300	49,900	89.6
1931	1,000	9,200	18,550	1,000	1,280	9,800	40,830	72.8
1932	700	6,000	15,190	1,000	1,050	7,700	31,640	56.1
1933	600	6,100	13,930	900	960	7,900	31,390*	55.4
1934								65.0E

* 1 billion dollars added because of incompleteness in 1933 data.

The residuals in the above examples appear to be mostly systematic, with but little variable element, which is the justification for interpolating. Should cases be found where the residuals appear to include considerable random elements, a combination of smoothing and interpolation would be called for.[9]

APPENDIX B.

Method of Estimating Weighted National Income

Table 6 shows how the index of a weighted national income series

[9] A paper dealing at length with the interpolation of time series is now in preparation.

was constructed and the weighting given to the different types of income. The weights were based on the relative proportion spent for attire by the characteristic recipient of each type of income. These weights were selected after a study of *America's Capacity to Consume*, published by the Brookings Institution, and are as follows:

Income Series from "Capacity to Consume", 1918–1929	
	Weight
Wages	1
Pensions, etc.	1
Salaries	$\frac{1}{4}$
Entrepreneurial	1
Property Income	1/10
Income Series from "Survey of Current Business", January, 1935	
Salaries, heavy industries	$\frac{1}{4}$
Wages	1
Salaries and wages, other industries	7/10
Compensation, etc.	1
Dividends and interest	1/10
Individual Enterprisers	1

The income for 1921 and 1927 was lowered because it is believed that King's estimate for those years do not give proper weight to the amount of unemployment existing. The figure for 1933 was raised by about $1,000,000,000 because the Department of Commerce series for national income wrongly disregards farm benefits and emergency income.[10] Further study could doubtless improve this national income estimate greatly.

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[10] The Department of Commerce estimates of National Income have recently (August, 1935) been revised and do take into account farm benefits and emergency income.

COMMENTS ON THE MACRODYNAMIC THEORY OF BUSINESS CYCLES

By M. KALECKI

CERTAIN questions have arisen[1] concerning my macrodynamic theory of business cycles[2] which I consider of sufficient importance to warrant a detailed answer. I also wish to complete some parts of my original study which I think were presented too briefly.

1. Tinbergen makes the statement concerning my original article that, "remarkably enough, prices do not appear at all in the theory."[3] It can be easily shown that in reality my basic equation implies the dependence of investment activity on the ratio of prices to wages. My basic equation was[4]

$$I/K = f(B/K, p), \tag{1}$$

where I = the volume of investment orders per unit of time, K = the volume of existing industrial equipment, B = the total real income of capitalists, including amortization, p = the interest rate, and f is an increasing function of B/K and a decreasing function of p. Let P equal the real national income; it will also be a measure of the total volume of production. We can then write an identity,

$$\frac{B}{K} = \frac{B}{P} \cdot \frac{P}{K},$$

Where the ratio B/P is the capitalists' share of the national income, while P/K is proportional to the degree of use of industrial equipment. Both of these factors are an increasing function of the ratio of prices to wages, for as this ratio increases so does the share to capitalists; also, the use of industrial equipment will increase as it becomes remunerative to employ that part of the equipment previously "standing by" and attract labourers from the reserve of unemployment. The ratio B/K, therefore, will be an increasing function of the ratio of prices to wages, and equation (1) shows that the intensity of investment ac-

[1] R. Frisch and H. Holme, "The Characteristic Solutions of a Mixed Difference and Differential Equation Occurring in Economic Dynamics," ECONOMETRICA, Vol. III, 1935.

J. Tinbergen, "Annual Survey: Suggestions on Quantitative Business Cycle Theory," ECONOMETRICA, Vol. III, 1935, pp. 268–270.

[2] "A Macrodynamic Theory of Business Cycles," ECONOMETRICA, Vol. III, 1935.

[3] *Op. cit.*

[4] *Op. cit.*, pp. 330, eq. (9).

tivity, I/K, is an increasing function of the price-wage ratio and a decreasing function of the interest rate, p.

2. It may be asked why equation (1) does not contain the ratio of prices to wages directly. The reason is that only by using the ratio B/K was it possible to base the system on a very small number of variables.

By making certain assumptions concerning the components of the real income of capitalists we obtain[5] from equation (1)

$$\frac{I}{K} = f\left(\frac{1}{1-\lambda} \cdot \frac{C_1 + A}{K}, p\right), \tag{2}$$

where C_1 and λ are constants such that $C_1 > 0$ and $1 > \lambda > 0$ and A is the output of investment goods. We have, in this equation, three functions of the time, namely, I, K, and A, between which there exist two interrelations which result from the technique of production. Equation (2) also includes the rate of interest, p. In the next section it will be shown how p may be eliminated.

A study of equation (2) shows that it directly involves the cumulative character of the business ups and downs. Let us suppose, for instance, that at a certain moment we get from equation (2) a value of I greater than A, i.e., there are more investment orders than investment goods produced. It is clear that after a certain time the output of investment goods will increase, but, according to equation (2), I will again increase, this in turn entailing a further increase of A. In other words, if at a certain moment $I > A$, we have a self-winding-up process of expansion of investment activity.

Of course, the mere fact that the process is here described by means of only two variables does not mean that the process itself is going on somewhere outside the range of price movements and changes in total production. If at a certain moment I is greater than A, and as a result the output of investment goods begins to grow over the preceding level L, that growth must be financed by creating purchasing power. We have then the so-called forced saving, that is to say, prices rise so much in respect to wages and with it the total volume of production that capitalists additionally "save" from their increased profits a money-equivalent of increment in output of investment goods. In that way, during the above process of cumulation of investment activities, the ratio of prices to wages is steadily increasing, and so is the total volume of production. The advance of prices is the very link uniting the increase of A to that of I; while increased investment orders lead directly to an increased output of investment goods, the increased out-

[5] *Op. cit.*, pp. 327–331.

put of investment goods, accompanied by forced saving, stimulates investment activities indirectly through the advance in prices which it calls forth. During a business recession these processes are reversed.

3. In the above description of the cumulative process, changes in interest rate were left out of consideration. As prices and production increase, the demand for means of payment becomes greater as well, and, as a result, the interest rate must rise. Thus, during the upward cumulation in equation (2), both arguments are simultaneously rising, and the rise of each of these factors affects the intensity of investment activity, I/K, in an opposite direction. If both of these influences were exactly equal, there could, of course, be no cumulative rise of investment activity and its intensity would remain stable; we then would have to do with the case of the so-called "neutral money," when any creation of purchasing power is impossible and there are no cyclical fluctuations. But in real life the credit system is elastic enough to prevent such a rise in p as would offset the stimulating influence of the higher ratio of prices to wages on investment activity. We can then say

$$\frac{I}{K} = \phi\left(\frac{C_1 + A}{K}\right), \tag{3}$$

where ϕ is an *increasing* function. The last condition, as may be seen from the above argument, means nothing else but that we have to deal with an economic system with "non-neutral" money, i.e., a system under which purchasing power may be created. The more elastic the credit system, the greater the rate of growth of the function ϕ, *caeteris paribus*. That statement will be of importance when I come to considering the questions of Frisch and Holme.

4. From the fact that the output of investment goods, A, comes in the numerator of the right-hand side of the equation, it follows that business cycles must have a cumulative character. But the presence of K, representing the volume of industrial equipment, in the denominator indicates the source of hampering influences, putting a brake on cumulative processes. Indeed, when investment activity crosses the level required by replacement of worn out equipment, K begins to rise, and this hampers the growth of the expression $(C_1+A)/K$ and, subsequently, also the rise of I/K. It follows from the above that $(C_1+A)/K$ is an increasing function of the ratio of prices to wages. The fact of the above fraction slowing the rate of growth because of the rise of K actually means that creation of purchasing power calls forth a stronger rise of prices in respect to wages in the case where the volume of industrial equipment is constant than in the case where it is expanding.

In the first instance the demand is met only by employing those parts of existing industrial equipment which are producing at higher cost; in the second, also new plants are constructed. The hampering influence of the rise of K may[6] lead to a state when investment orders I, after crossing their high point, begin to fall, and thus start a downward cumulation. In an analogous way, under my business cycle theory, the end of the cyclical drop (and the following rise) occurs as a result of contraction of industrial equipment.

Tinbergen is rather sceptical in respect to the part attributed in my theory to changes in volume of industrial equipment. I think that the process described above of hampering the upward cumulative movement, and the eventual slump, may well be regarded as a more precise definition of what common sense denotes as "over-investment" during business prosperity.

5. Assuming that ϕ is a linear function, we may say

$$I = m(C_1 + A) - nK. \tag{4}$$

We have shown in Section (3) that the rate of growth of the function ϕ is more rapid, the greater the elasticity of the credit system, i.e., the smaller the advance of the interest rate called forth by the given rise of prices and production; thus, the coefficient, m, is larger the greater the elasticity of the credit system in the above sense.

Equation (4), together with the interdependence of the variables I, A, and K, resulting from the technique of production,[7] make it possible to state a mixed difference and differential equation for I as function of t. The solution of that equation shows that I exhibits harmonic oscillations with a constant, increasing, or decreasing amplitude. The period of oscillation, and the rate of progression and degression of amplitude depend on the empirical coefficients m and n and on the duration of the "gestation period" of investments.

I have obtained for m and n the numerical values: $m = 0.95$, $n = 0.12$, and 10 years as the duration of the cycle, assuming constant amplitude of fluctuations. This assumption was based on the fact that in real life this amplitude does not exhibit any steady progression or degression.

Frisch and Holme object to the above assumption of constant amplitude. They are right, for it is by no means sufficient to say that an assumption is correct just because it is confirmed by the conditions of real life. It must be made clear why the real life is like that, otherwise the particular predilection it shows for a constant amplitude might

[6] "May," but not "must," unless there are some definite conditions fulfilled by the function. Cf. Kalecki, *op. cit.*, p. 335.

[7] *Op. cit.*, pp. 328–329, eqs. (4), (7), and (8).

appear metaphysical. I shall try to give, in a summary way, a hypothetical explanation of that state of affairs without pretending to have found a definite solution of that very complex problem.

Let us suppose that m has a slightly smaller value than that given above; it is easily seen that this results in damped oscillations and in a short time the business cycle will practically disappear. But the requirements of liquidity of banks and enterprises will become less stringent and the disappearance of cyclical fluctuations will have the effect of an increase in reserves. The credit system will become more elastic and a given rise of price and production will call forth a less marked advance in the rate of interest.

As we have just shown, the more elastic the credit system, the greater m will become, and, therefore, the damping of oscillations will lead to an increase in m and thus create a tendency towards return to fluctuations with a constant amplitude.

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VERTICAL AND HORIZONTAL SHIFTS IN DEMAND CURVES[1]

By Geoffrey Shepherd

A few years ago an article appeared in which a distinction was made between vertical and horizontal shifts in demand curves.[2] Two years later, this distinction was discussed by a colleague of mine,[3] and last year his comments were discussed in this journal by a third seeker after truth.[4] The subject is new and appears to be fruitful, and our views are not entirely unanimous; a restatement of the matter may clarify it and lead to further helpful discussion.

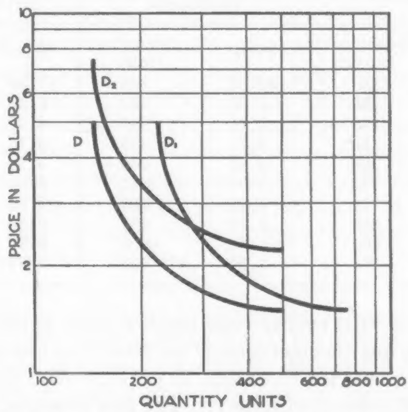

Figure 1. Vertical and Horizontal Shifts in a Concave Demand Curve.

The fundamental idea of vertical and horizontal shifts in demand curves is simple. We can deal with it best by starting with the concept of the demand schedule. A typical demand schedule is shown in Table 1, section A.

[1] Journal Paper No. J-341 of the Iowa Agricultural Experiment Station. Project No. 462.

[2] Shepherd, G. S., "Supply and Production, Demand and Consumption," *Journal of Farm Economics*, Oct. 1931, pp. 639–642.

[3] Thomsen, F. L., " 'Vertical' and 'Horizontal' Shifts of Demand," *Journal of Farm Economics*, July 1933, pp. 567–570. See also G. S. Shepherd, "Vertical and Horizontal Shifts in Demand Curves," same journal, Oct. 1933, pp. 723–729.

[4] Hartkemeier, H. P., "Note on Shifts in Demand and Supply Curves," Econometrica, Oct. 1935, pp. 428–434.

The demand curve D based on these figures is shown in Figure 1. Both the vertical and horizontal scales in the chart are logarithmic. This preserves parallelism in the curves throughout the various shifts in their position that are considered. The reasoning, however, is independent of the kinds of scales used.

The use of a curved demand line on a logarithmic scale will help to bring out the point more clearly than the use of a straight line, though the reasoning in both cases is the same. Either a concave or a convex curve may be used. We shall start with the former.

TABLE 1.—DEMAND SCHEDULES

A. Original Demand Schedule		B. Population Increased 50% Purchasing Power Unchanged		C. Purchasing Power Increased Population Unchanged	
Price	Quantity Units	Price	Quantity Units	Price	Quantity Units
$5.00	150	$5.00	225	$7.50	150
4.00	155	4.00	232	6.00	155
3.00	170	3.00	255	4.50	170
2.50	190	2.50	285	3.75	190
2.35	200	2.35	300	3.52	200
2.00	235	2.00	352	3.00	235
1.90	250	1.90	375	2.85	250
1.70	300	1.70	450	2.55	300
1.55	400	1.55	600	2.32	400
1.50	500	1.50	750	2.25	500

Suppose, now, that twenty years elapse and the population consuming the good in question increases 50 per cent. If no changes have taken place in the demand per consumer, 50 per cent more goods could now be sold at each price than formerly. The new situation is represented in Table 1, Section B, in which each quantity figure is 50 per cent higher than the corresponding figure in the left hand section. The price figures remain unchanged. The new curve, D_1, is shown in Figure 1.

Now let us suppose that, instead of the population increasing 50 per cent, it had remained unchanged, but the purchasing power of each consumer had increased. No other change in demand took place, but, because of their increased purchasing power, consumers were willing to pay, let us say, 50 per cent more for each quantity than formerly. This situation is shown in Table 1, section C, where each price figure is 50 per cent higher than the corresponding figure in section A, the quantity figures remaining unchanged. The new curve, D_2, is shown in Figure 1.

A concrete illustration of this sort of change in demand is a rise or decline of the general price level. This represents a change in the amounts of money which consumers would offer for the same amounts

of goods as before. Another illustration is the effect of the AAA processing tax on hogs, which shifted the demand curve for hogs downwards by the amount of the tax.

The curve D_1 is an illustration of a horizontal shift in the position of the demand curve. The other curve, D_2, is an illustration of an equal vertical shift. The difference between the two curves seems clear.

EFFECT UPON PRICE PAID AND QUANTITY TAKEN

One might think that a vertical upward shift in the demand curve would result in a higher price being paid for the same quantity of goods as before, and that a shift to the right in the demand curve would result in more goods being sold at the same price as formerly. Conversely, one might reason backwards from the changes in quantity or price, and say that if the price had increased while the quantity taken remained unchanged, the demand curve must have shifted upwards. But this would be wrong. Production and price simply represent the intersection point of a demand and supply schedule. The effect of a horizontal or of a vertical shift in a demand curve depends upon the supply curve as well as upon the demand curve. Whether a shift in the location of a demand curve, either upwards or to the right, will result in an increase in the price or in the quantity taken, or both, depends upon the conditions of supply; that is, upon the slope of the supply curve and changes in its location.

Under conditions of constant costs, for example, a vertical rise in the demand curve would result in an increase, not in the price for the same quantity, but in the quantity taken at the same price. Conversely with a fixed stock of a good, a horizontal shift to the right in the demand curve would result, not in an increase in quantity taken at the same price, but in an increased price paid for the same quantity. The nature of the supply curve, and shifts that may have taken place in its location, determines the proportion in which an increase in demand, either upward or to the right, is expressed as an increase in the price or in the quantity taken.

This point is illustrated in Figure 2. In section A of this figure, a convex demand curve is shown shifting to the right. The supply curve, however, happens to be inelastic (fixed stock). As a result, although the demand curve has moved to the right, the intersection point of the demand and supply curves has necessarily (because of the inelasticity of the supply curve) moved upward. A higher price is paid for the same quantity as before.

In section B of Figure 2, the opposite situation is shown. The demand curve shifts upward, but the supply curve happens to be fully elastic (constant costs). The result of the upward shift in the demand

curve is a shift of the intersection point to the right; that is, more goods are taken at the same price as before.

Two things, therefore, are evident. (1) Whenever the demand curve is convex or concave the demand curve as a whole is different in its position after a vertical and after a horizontal shift, and (2) the effect of a shift in demand upon price paid and quantity taken depends, not upon the direction of the shift, but upon the nature of the supply curve.

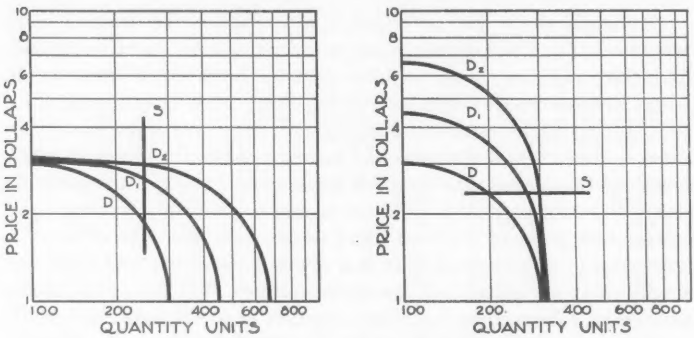

FIGURE 2. Effect of Elasticity of Supply Curve.

STRAIGHT-LINE DEMAND CURVES

We come now to the consideration of straight-line demand curves. Before beginning, we must decide whether we mean straight line curves on arithmetic paper or on double logarithmic paper. Practically all the statistical price-quantity curves published in recent articles and bulletins are drawn on arithmetic paper. But the concept of changes in demand is fundamentally proportional in character, and changes in demand in actual life are usually[5] proportional. A proportional change in demand shown on arithmetic paper results in a new demand curve that is not parallel with the old. If a proportional change in demand is shown on double logarithmic paper, however, the new demand curve remains parallel with the old. Perhaps the best plan here is to consider separately both arithmetic and logarithmic straight-line demand curves.

ARITHMETIC SCALES

Let us first consider straight line curves on an arithmetic scale.

In this case the difference between the curves resulting from a hori-

[5] In certain cases a change in demand may be arithmetic, A change in distributor's margins, for example, results in a vertical arithmetic shift.

zontal and from a vertical shift of 50 per cent is evident, not only when elastic and inelastic curves are used, but also when an intermediate curve with slope of -1 is used. This is shown in Figure 3. The elasticity of the curves remains unaffected, since the changes in demand are proportional changes, but the slope of the curves is altered.

LOGARITHMIC SCALES

If elastic or inelastic straight line curves on *logarithmic* scales are used, the position of the curve after a 50 per cent upward shift will be different from its position after a 50 per cent shift to the right, and so will the price paid and quantity taken.

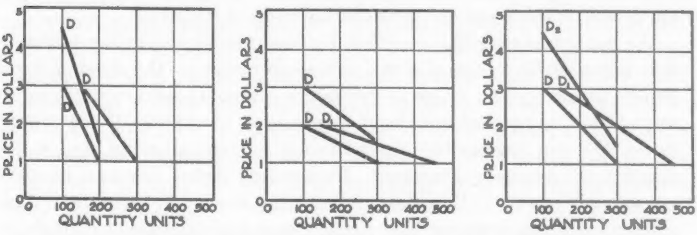

FIGURE 3. Shifts in Straight Line Demand Curves. Arithmetic Scales.

It is only in the rare case of a straight line demand curve on a double logarithmic scale, with a slope of -1 throughout, that the position of the curve would be the same after either shift. In this case the effect of a horizontal shift in demand upon the location of the intersection point, that is, upon production and price, would be identical with that of an equal vertical shift. After population has increased 50 per cent, consumers as a group might either pay higher prices for the same quantity as before, or take larger quantities at the same price as before, or some intermediate combination of the two, according to the nature of the supply curve. If the supply curve were a vertical straight line (fixed stock), the consumers would pay more for the same quantity. If the supply curve were a horizontal line (constant costs), they would take a larger quantity at the same price. If the supply curve had a slope intermediate between vertical and horizontal, the effect on price and quantity would be intermediate—both price and quantity would increase, in proportions determined by the slope of the supply curve.

REASON FOR DISTINCTION

The reason for distinguishing between vertical and horizontal shifts in demand curves is this: we start with a price series, and find that it

fluctuates. What is the reason for the fluctuations? The accepted procedure among economists is to group the possible causes under the two heads, demand and supply. Investigating these two groups, we come to the conclusion that the demand changed, or the supply changed, or both. If our objective is to reduce price fluctuations in the future, we know then whether we need to concentrate our attention upon changes in demand, or in supply, or in both.

The purpose of the distinction between horizontal and vertical shifts in demand (or supply) curves is to enable us to carry our investigation one step further. We have determined, let us say, that the chief cause of the price fluctuations was the changes that took place in demand. The demand curve shifted, and the question is, which way did it shift—up, down, or sideways, or some combination of these?

We cannot answer this question by observing whether the intersection point *or the range of actual price-experience on the demand curve* (which is merely the range of intersection points) shifted up or sideways. That, as pointed out above, depends on the nature of the supply curve. We can answer the question only by remembering that a demand curve represents demand. Economists define demand as consumer's willingness to buy certain quantities at certain prices; and that willingness exists whether the supply curve has fluctuated enough to reveal it in actual transactions or not. This means that the demand curve extends both ways, beyond the range of past experience in the market—ultimately, until it cuts the vertical and horizontal axes where quantity and price respectively are zero. (The curve will not extend indefinitely; it will cut both axes at some finite points.)[6]

We can tell which way the demand curve has shifted, then, by going behind the original price and quantity data on which the demand curve rests. We are seeking to explain why the data changed. It seems obvious that, if population increased 50 per cent and if no other important change took place, the curve moved to the right, not upwards and to the right. The question can be demonstrated statistically when the demand curve is strongly curved, when, for example, it is a convex curve that cuts both axes at almost right angles, or a sloping straight line that flattens out or gets steeper as either axis is approached; it is

[6] Usually these points, like those shown in Figure 3, will represent prices or quantities not greatly (say 100 per cent) in excess of the highest prices or quantities that have been actually experienced in the market, unless the demand is extremely inelastic, as for salt or water, or extremely elastic, as for human foods that can be fed to livestock if produced in excess. Substitution of other products "levels off" most demand curves as they approach the vertical axis, and rapidly declining marginal utility with increasing quantity causes most demand curves to cut the horizontal axis at a point not very far out to the right.

equally true, only less obvious, when a straight line is used. We are on logically sounder ground in endeavoring to carry our explanation of price movements down to its ultimate causes if we recognize that a demand curve may shift either horizontally or vertically, or some combination of both, independent of which way (if any) the intersection point or the range of intersection points moved. We need to investigate what happened to the demand curve first, and then turn to a study of what happened to the supply curve; for movements in demand curves and supply curves (except in a roundabout sense, as during inflation or deflation) are independent of one another.

We are not studying movements in the intersection points of demand and supply curves; if we were, we would be studying only movements in production and prices; what we are trying to do is to study the movement of demand and supply curves that lie behind and cause these movements in prices and production. Economic theory has provided the research worker with conceptual tools for analyzing movements in prices and production into changes in demand and supply, that is, into movements of demand and supply curves. It seems to me that we are ready now to take the next step and analyze these movements into their horizontal and/or vertical components. We are expanding our data concerning population, incomes, pay rolls, wage rate indexes, general price levels, distributive margins and so forth, which should enable us to carry our economic analysis this one step farther and give these questions a quantitative answer.

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ANNUAL SURVEY OF STATISTICAL TECHNIQUES: THE CORRELATION AND ANALYSIS OF TIME SERIES—PART II

By Charles F. Roos

I. THE CORRELATION OF TIME SERIES

1. *Introduction.*—When each correlation variable becomes a function of time, as must be the case in economic measurements, various difficulties are encountered. First, there is the question of how long a time interval should be chosen in order that the observations may be regarded as independent. Even when this problem is solved, there still remains the question of trends, insofar as these involve movements which are common to the dependent variable, and one or more of the independent variables, but are not causal. Also a "proper" function must be chosen. If these questions are satisfactorily answered, the investigator may proceed to determine correlation coefficients, but what, if anything, do these mean after they are obtained? These are the fundamental problems to be discussed in this paper, the treatment being selective rather than exhaustive.

2. *Independence of Observations.*—In analyzing time series observers have usually assumed that there are two kinds of variations:

(a) Systematic variation: for instance, (1) secular variation, (2) periodic variation such as the seasonal variation in economic phenomena and (3) cyclic variation (business cycles) which is not strictly periodic.

(b) Random or chance variation: for instance, the residuals left after graduation of the observations by seasonal analysis and a good smoothing formula. The theory and treatment alike have assumed that the random errors are the same kind of thing as the random errors of the classical theory of observations, developed for measurements on precision instruments.

In the theory of observations, the residual v_1 in the observation y_1 and the error e_1 in that observation are conceived to be the net result of a multitude of individually small shocks. Each such minute shock is an "elementary error," and their summation is the actual error. Similarly with v_2 and e_2 in the observation y_2; but there are no shocks common to e_1 and e_2. For all practical purposes they are wholly independent of one another.

Although in many economic series residuals arise in part from causes of the same type as the random variations just mentioned, there are additional errors which arise because two successive errors have the

same "elementary error" but are opposite in sign. Thus, in the case of building contracts a particular contract may be "due" to be signed on May 31 and reported in that month; but "by chance" it is recorded on June 1 and goes into that month. The May observation is of the form $B_1 - e$ and that of June $B_2 + e$, one being increased and the other decreased by the same amount. Where consumption of a commodity is determined by adding production and decrease in inventory, errors in the inventory figure through failure to measure it correctly or to measure it at the proper time will enter the succeeding determination with opposite sign. Errors and residuals of this type are referred to by Victor S. von Szeliski as *alternating residuals and alternating errors.*[1]

Let $y_i (i = 1, 2, \cdots, n)$ be observed values composed of true values x_i, alternating errors a_{i-1}, a_i, and random errors e_i. If the y_i are summed, it follows that

$$\begin{aligned}\sum_{i=1}^{n} y_i &= \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} (a_{i-1} - a_i) + \sum_{i=1}^{n} e_i \\ &= \sum_{i=1}^{n} x_i + a_0 - a_n + \sum_{i=1}^{n} e_i.\end{aligned}$$

Thus, if monthly observations are summed over a period alternating errors cancel out with the exception of the errors in the terminal values. On the other hand, if the interval of time between successive observations is "too small," the alternating error will be so large relative to the observation that the series will appear to be random; for example, a daily, weekly or even monthly series of building contracts awarded appears to be a completely random series, whereas a series of annual totals exhibits comparatively smooth cycles with very little of the random element. While alternating errors still exist at each end of the larger time interval, the size of the error is reduced relative to the size of the observation. This naturally suggests the following question:

Question 1. What is the largest permissible alternating error relative to the observation, i.e., max. a_i/min. y_i, if the observations are to be regarded as independent within reasonable limits of error, and what are these limits?

Until this question is answered, it is of doubtful value to calculate probable errors of coefficients of correlation between time series. In fact, until it is settled, correlation coefficients of time series must be looked upon with a great deal of skepticism.[2] To demonstrate more clearly the

[1] See C. F. Roos, *Dynamic Economics*, 1934, Appendix II.

[2] It is perhaps in order to say that one of the most famous examples of correlation coefficients which went awry because of large alternating errors over a short period of time is the correlation between wholesale prices and the price of gold.

point involved consider x and y defined by the observations in time t, below:

TABLE I

	True Values		Alternating Errors		Observed Values	
t	$x(t)$	$y(t)$	$x(t)$	$y(t)$	x	y
1	1	2	+1	+1	2	3
2	2	4	−1	−1	1	3
3	3	6	+1	+1	4	7
4	4	8	−1	−1	3	7

For this set of data, if we use four intervals of time as given, the line of best fit (y independent) is

$$y = 1.6x - 1;$$

whereas if we choose an interval of time twice as long we obtain the different regression

$$y = 2x.$$

Another type of interrelationship of observations, called "serial correlation" by G. U. Yule, "Gliedkorrelationen" by Oskar Anderson and "Korrelationsfunktionen" by Eugene Slutsky, arises whenever any observed value is correlated, say, to an extent ρ_1 with the preceding.[3] In general, the correlation with the nth preceding one will be ρ_1^n and as Bartlett[4] shows, if we correlate two series x and y having serial correlations of ρ_1 and ρ_2 respectively, we obtain the expected value

In his studies Professor George F. Warren used a year as the unit of time whereas the Roosevelt Administration attempted to interpret the correlations as applying to daily or weekly periods.

[3] G. U. Yule, "Why Do We Sometimes Get Nonsense Correlations?" *Jour. Royal Stat. Soc.* 1926, p. 14; Oskar Anderson, "Ueber die Anwendung der Differenzenmethode ('Variate Difference Method') bei Reihenausgleichungen, Stabilitätsuntersuchungen und Korrelationsmessungen," *Biometrika*, Vol. 19, 1927, pp. 53–86, and "Die Korrelationsrechnung in der Konjunkturforschung," *Frankfurter Gesellschaft für Konjunkturforschung*, Vol. 4, (1929); and Eugene Slutsky, "The Summation of Random Causes as the Source of Cyclic Processes," *Problems of Economic Conditions*, Vol. 3, No. 1, Conjuncture Institute, Moscow, 1927. An English translation of the Russian article by Slutsky will appear shortly in *Econometrica*. See also Holbrook Working and Harold Hotelling, "The Application of the Theory of Error to the Interpretation of Trends," *Proc. Amer. Stat. Ass'n.*, March 1929, pp. 73–85; and E. C. Rhodes, *Jour. Royal Stat. Soc.*, Vol. 90, 1927.

[4] M. S. Bartlett, "Some Aspects of the Time Correlation Problem in Regard to Tests of Significance," *Jour. Royal Stat. Soc.*, Vol. 98, (1935) pp. 536–543. Also by the same author, *Proc. Roy. Soc. Edin.*, Vol. 53, (1932–33), pp. 260–283.

$$E\left\{\frac{1}{n^2}\left(\sum x_r y_r\right)^2\right\} = \frac{1}{n}\sigma_1{}^2\sigma_2{}^2\left(1 + \frac{2\rho_1\rho_2}{1 - \rho_1\rho_2}\right)$$

approximately, whence the variance of the sample correlation between the two series is $\frac{1}{n}\left(\frac{1+\rho_1\rho_2}{1-\rho_1\rho_2}\right)$ approximately, indicating less accuracy in estimating our correlation when ρ_1 and ρ_2 have the same sign. Thus, as Bartlett states, if $\rho_1 = \rho_2 = .6$, "our accuracy in estimating a correlation that is really zero, or near zero, would be equivalent to that obtainable with less than half the same number of independent observations." Bartlett then shows that if the two series x and y are uncorrelated in any way with each other, the expected value

$$E\left\{\frac{1}{n^2}\sum x_r y_r \sum x_r y_{r+1}\right\} = \frac{1}{n}\sigma_1{}^2\sigma_2{}^2\left(\frac{\rho_1 + \rho_2}{1 - \rho_1\rho_2}\right)$$

approximately, whence the correlation of two successive correlations is

$$\left(\frac{\rho_1 + \rho_2}{1 - \rho_1\rho_2}\right) \Big/ \left(\frac{1 + \rho_1\rho_2}{1 - \rho_1\rho_2}\right) = \frac{\rho_1 + \rho_2}{1 + \rho_1\rho_2}$$

approximately, and for $\rho_1 = \rho_2 = .6$ this would be .88.

At this point we must urge strongly that the key to the fundamental problem of correlating time series, namely, the determination of the number of independent observations, will be found associated with the questions of alternating and serial errors.

Suppose for instance, that we have two time series with the following structure

$$X\begin{cases} x_i = X_i + b_i + f_i + E_i \\ x_{i+1} = X_{i+1} - b_i + f_{i+1} + E_{i+1} \end{cases}$$
$$Y\begin{cases} y_i = Y_i + a_i + e_i + \epsilon_i \\ y_{i+1} = Y_{i+1} - a_i + e_{i+1} + \epsilon_{i+1} \end{cases}$$

where X_i and Y_i are, what we might call, underlying or fundamental values of observations x and y and are such that X_i and $X_{i\pm n}$ and Y_i and $Y_{i\pm n}$, $(n \neq 0)$, are uncorrelated; a_i and b_i are alternating errors; e_i and f_i are serially correlated errors; and ϵ_i and E_i are purely random errors having unassignable causes.

Then if we correlate y_i and y_{i+1} (or x_i and x_{i+1}), we will get a value either greater than or less than the coefficient of serial correlation depending upon the signs of the alternating errors. Suppose for definite-

ness that the serial correlation is lowered, say, almost to zero. Even so the observations having such subranges are not independent; for if we combine observations, say, sum successive ones, we will reduce the size of the alternating errors relative to Y_i and e_i and E_i, and a lag correlation may then conceivably raise the serial correlation. Similarly a further increase in the subrange may still further increase the serial correlation, but a limit must be reached at which the lag correlation will begin to decrease. And a decline to zero correlation will frequently occur, and in such event we would be reasonably safe in concluding that observations having such subranges are independent.

It may be added that the Cowles Commission in an unpublished study has demonstrated the existence and persistence over long periods of such structure in stock price indexes.

3. *Selection of Trend.*—Once the question of choice of interval of time is settled, it usually becomes necessary to decide what to do with the trends which are so often common to the variables to be correlated. Presence of similar trends in two variables x and y to be correlated will frequently result in a high correlation even when, in no sense of the word, is the trend in x the cause of the trend in y or vice versa, the two trends rather being due to a common third cause or pure accident.

Some statistical workers, inspired by various papers of Warren M. Persons and others of the Harvard School have fallen into the[5] point of view maintaining that one should always "eliminate trend" and correlate residuals. A reference to a physical problem is probably the quickest way to show the danger of following this procedure blindly. Let V = voltage, R = resistance and I = current, so that $V=RI$. It is possible to vary V so that $V=3Rt+$random errors; and obviously, to remove the trend $3\,Rt$ from V and $3\,t$ from I and correlate the residuals, would be to correlate random errors. In the case of economic time series, the relative size of the error is usually much larger than in the physical example just given. Thus, it may happen that the removal of a linear or a quadratic trend will leave little more than random errors. However, even here we can still say that the two series are related, the trends themselves defining the relationship.

In more detail suppose that $\phi_1, \phi_2, \cdots,$ are Tchebycheff polynomials of time t and that trends

$$Y = C_0 + C_1\phi_1 + \cdots + C_q\phi_q, \tag{2}$$

$$U = K_0 + K_1\phi_1 + \cdots + K_q\phi_q, \tag{3}$$

where $C_0, \cdots, C_q$ and $K_0, \cdots, K_q$ are constants, are fitted to the series $y(t)$ and $u(t)$, respectively. Let $z=y-Y$ and $x=u-U$. Then, as

[5] W. M. Persons, "The Correlation of Time Series," Chapter X, *Handbook of Mathematical Statistic*, 1924.

Charles Jordan has shown,[6] the simple correlation between z and x is given by the formula

$$r = \frac{\frac{1}{n}\sum yu - C_0K_0 - C_1K_1 \cdots - C_qK_q}{\sqrt{(\sigma_y{}^2 - C_1{}^2 - C_2{}^2 \cdots - C_q{}^2)(\sigma_u{}^2 - K_1{}^2 - K_2{}^2 \cdots - K_q{}^2)}}$$

where σ is the standard deviation of y, and n is the number of observations in the series. If r is zero within random errors, the relationship between y and U must be contained in the two expansions (2) and (3). For instance, if in (2) all coefficients C_i after C_1 are not statistically significant, we have $Y = C_0 + C_1\phi_1$, where ϕ_1 is linear in t, and hence, since

$$t = aY + b,$$

the relationship between U and Y is given by

$$U = K_0 + K_1\phi_1(ay + b) + K_2\phi_2(ay + b),$$
$$+ \cdots + K_q\phi_q(ay + b),$$

where K_q is the last significant constant. A similar analysis may be carried through for general orthogonal functions instead of the orthogonal polynomials which Jordan uses. The form of the correlation coefficient will be the same, but its value will depend on the particular orthogonal functions used.

And the probabilities of statistical significance of regression coefficients do not help much, if at all, in determining the true relationships. In fact, the author has shown[7] that, by using one set of orthogonal functions whose regression coefficients are statistically significant, a correlation of almost -1.00 can be obtained between the residuals from trend of two series; and by using different orthogonal functions with coefficient of almost $+1.00$ can be obtained. Still other choices of orthogonal functions will give any value between that is desired. The difficulty seems to arise because the probability tests of significance do not take into consideration the choice of orthogonal functions. Therefore, the use of the coefficient of correlation r to indicate a linear relationship between the two functions considered, is open to considerable suspicion; but the literature seems to be chiefly concerned with giving methods for determining the coefficient with no discussion of its use or significance with respect to the choice of function.

[6] Charles Jordan, "Sur la Determination de la Tendance Seculaire des Grandeurs Statistiques par la Methode des Moindres Carrés." Triage àpart de *Journal de la Societé Hongroise de Statistique*, 1929, No. 4. See also, A. C. Aitken, *Proc. Roy. Soc. Edin.*, Vol. 53 (1932), pp. 54–78.

[7] *Dynamic Economics*, Appendix I, pp. 246–250.

4. *The Variate Difference Method.*—An alternative to the fitting of trends is the well-known variate difference method.[8] If we suppose

$$x = \phi(t) + X$$
$$y = \psi(t) + Y,$$

where X and Y are the parts of x and y independent of the time t, then "Student" has shown (*Biometrika*, Vol. 10) that when m is sufficiently large

$$r_{\Delta^m x \Delta^m y} = r_{\Delta^{m+1} x \Delta^{m+1} y} = r_{xy}$$

and Oskar Anderson has derived the standard errors (*Biometrika*, Vol. 10). However, as Cave and Pearson have observed in applying these methods to economic data, we frequently are compelled to take $n>6$ and thus, if our series are short, as is so often the case, we are unable to reach definite conclusions. More recently Bartlett has shown[9] that for no real correlation between two series, we have

$$E\left\{\frac{1}{n^2}\left[\sum (x_r - x_{r-1})(y_r - y_{r-1})\right]^2\right\} = 6\sigma_1{}^2\sigma_2{}^2/n$$

approximately or for the variance of the correlation coefficient $3/2n$ instead of $1/n$. In other words, in the estimation of a coefficient near zero one-third of the information would be discarded by the use of first differences. More generally, the efficiency is given by the following table:

TABLE II

Differences	0	1	2	3	4	5
Efficiency	100	66.7	51.4	43.3	38.1	34.4

It is thus seen that as we proceed with high order differences, the sacrifice of information is indeed large. Moreover, R. A. Fisher's remarks following a paper by Yule[10] show that, when lag correlations

[8] This method was discovered independently by Miss F. E. Cave in 1904 and R. H. Hooker in 1905. "Student" extended it to the non-linear case in 1910 and Oskar Anderson derived the probable errors. For references, a discussion of contributions, and early applications, see Beatrice M. Cave and Karl Pearson, "Numerical Illustrations of the Variate Difference Correlation Method," *Biometrika*, Vol. 10, 1914, pp. 341–355.

[9] M. S. Bartlett, *op. cit.*, p. 540. See also W. M. Persons, "The Correlation of Time Series," Chapter X, *Handbook of Mathematical Statistic*, 1924.

[10] G. U. Yule, "The Problem of Time Correlation with Special Reference to the Variate Difference Correlation Method," *Jour. Royal Stat. Soc.*, *Vol. 84*, (1921), pp. 497–576.

For discussions of difficulties arising from lag correlation see Warren M. Persons, "On the Variate Difference Correlation Method and Curve-Fitting,"

exist, it is often impossible to know what is meant by a correlation obtained by the variate difference method. Therefore, in view of these considerations the technique must be used with considerable caution; but the method, if so used, is undoubtedly valid and useful.

5. *The Choice of Regression Function.*—In a very interesting paper[11] presented at the 1936 sessions of the Cowles Commission Seminars in Colorado Springs, Herbert E. Jones of the Commission shows how the regression function in the correlation analysis of time series depends upon the type or general form of the series being correlated. Deviations from an adequately fitted trend are used, and the cycles divided into two groups, that portion of the curve being above the trend, called the positive parts of the cycles, and that being below the trend called the negative parts. Jones classifies time series with respect to two general criteria—one a criterion of "steepness" and the second, a criterion of "skewness." The criterion of "steepness" is the coefficient of variation, i.e., the standard deviation divided by the mean, and signified by v. This coefficient is computed for both parts of the cycle, the coefficient for the positive part designated by v' and that for the negative part, v''.

Jones then shows that if two series, X and Y, are positively correlated, the ratio of their criteria of "steepness" indicates the general form of the regression function. If the ratio of v_y to v_x is taken, then: $v_y/v_x \gtreqless 1$ respectively, as the slope of the regression curve tends to increase, remain constant, or decrease. This ratio is computed for the positive and negative parts of the cycle and, when *both* are either greater than one or less than one, the regression is S shaped; when one ratio is greater than one but the other less than one, the regression tends toward a hyperbola or parabola; if the ratios equal one, the regression is linear. When the series are negatively correlated, the positive half cycles of one series must be compared with the negative half cycles of the other.

Quar. Pub. Amer. Stat. Assoc., June 1917; and Karl Pearson and Ethel M. Elderton, "On the Variate Difference Method," *Biometrika*, Vol. 14 (1923), pp. 281–310. Pearson and Elderton criticized the papers of Persons and Yule and derived formulae showing the effects of both short and long periodic terms on correlations of nth differences and on the variances of the series of differences.

For further discussion of the method and consideration of some of the criticisms see Oskar Anderson, "Die Korrelationsrechnung in der Konjunkturforschung," *Frankfurter Gesellschaft für Konjunkturforschung*, 1929, pp. 1–141; Gerhard Tintner, *Prices and the Trade Cycle*, 1935, especially pp. 9–27 and pp. 81–85; and Oskar Anderson, "On the Logic of the Decomposition of Statistical Series into Separate Components," *Jour. Royal Stat. Soc.*, Vol. 90 (1927), pp. 548–569.

[11] Herbert E. Jones, "The Nature of Regression Functions in the Correlation Analysis of Time Series."

The criterion of skewness, designated by Jones as j, is the distance from the centroid of the area under the cycles to the midpoint of the half cycle; it is negative if the centroid lies to the left of the midpoint and positive when it lies to the right. It is easy to see that, when $j_y - j_x \neq 0$, the regression curve will be in the form of a double loop, that is, resembling a figure 8 in shape. If the difference of the criteria of skewness is negative, the points on the regression curve travel in a clockwise direction, while if positive, they travel counterclockwise. If the difference in skewness is large, not one but two regression curves are necessary—one showing the relation during the upward swing of the cycle, the second the relation during the downward swing.

Another interesting part of Jones' analysis is that he readily demonstrates the theorem that if the series are out of phase, i.e., one series lags behind the other, an open loop will be formed in the scatter diagram when successive points are connected. If the series are positively correlated, a clockwise rotation in these loops means that the Y series precedes the X series, and counter-clockwise rotations mean the X series precedes. If the series are negatively correlated, these rotations are reversed. By this method the limits for the optimum lag can easily and quickly be found by connecting successive points in a scatter diagram, if there is a consistent rotation in one direction we lag one of the series according to the above rule until the rotation is reversed. When we find a position where the lagging of one more item is enough to reverse the rotation, then we know the optimum lag lies between these two points.[12] Interestingly enough Jones uses four applications of actual time correlations to show the practical use of such analysis, the lag being easily found by means of the scatter diagram and the correlation coefficient being greatly increased by using non-linear regression functions when the ratio of the criteria of steepness indicates curvature.

6. *Linear Correlation When the Dependent Variate is Subject to Error.* —When it is desired to fit a straight line $y = a_1x + b_1$ to $m > 2$ observed points (x_iy_i) the method usually employed is that of choosing a and b so that they minimize the sum of the squares of the residuals of the y_i, that is, $\sum(a_1x_i + b_1 - y_i)^2$ is a minimum. Two regression lines can always be obtained, one by minimizing the residuals in the y_i and the other the residuals in the x_i. The first assumes the observations in x to be perfect, and the second the observations in y. But in the case of time series of economic data both (or more generally all) variates are subject to error, and neither is the best line. Graphical curve-fitters have perhaps come closer to the "best line," since as the author has

[12] These last results bear some resemblance to those obtained by Ragnar Frisch, "Correlation and Scatter," *Nordic Statistical Journal*, Vol. I, pp. 36–102.

observed, those who express their variates in units of standard deviations and then choose equal scales tend to minimize normal distance.[13]

While this gives *a priori* a more satisfactory line, there are obviously cases in which some intermediate lines are better, and, moreover, the method of minimizing normal deviates is not independent of the coordinate system chosen. The author attacked the general problem in a lecture also presented at the 1936 sessions of the Cowles Commission Seminars.[14] Here it was shown that for the set of observations (x_i, y_i), the associated weights p_i and the line, $ax+by+c=0$, the most general function $\sum p_i f(x_i y_i a, b, c)$ which remains invariant under translation, homogenous strain and rotation is $\sum p_i(ax_i+by_i+c)^n$ where n is any number. For the quantity ax_i+by_i+c to represent a distance in the direction $\tan \alpha = \delta y_i/\delta x_i$ where δy_i and δx_i are the errors in y_i and x_i respectively, we must also have $\cos \alpha\delta a + \sin \alpha\delta b = 0$. When $n=2$ (least squares' assumption), the conditions of solution are

$$\sum p_i x_i(ax_i + by_i + c) \sin \alpha = \sum p_i y_i(ax_i + by_i + c) \cos \alpha$$
$$\sum p_i(ax_i + by_i + c) = 0.$$

The method generalizes readily to planes and hyperplanes and to functions expandable in series involving the parameters linearly.

Suppose, as an illustration, that we wish to find a regression between wholesale prices of cotton goods and farm prices of cotton. We know that, over such short intervals of time as given by Table III, Y, the price of cotton goods, is equal to a weighted sum of X, the price of cotton at the farm, Z the average hourly earnings of labor used in cotton manufacturing, and E which is composed of fixed charges and profits or erratic elements; thus

$$Y = a_0X + a_1Z + a_2E.$$

For the data of Table III, the simple correlation between Y and Z is

[13] The literature on analytical methods for minimizing normal deviates is large. The earliest paper seems to be by R. J. Adcock who wrote in the *Analyst*, Vol. 4, 1877. Later fundamental papers on the same subject have been presented by Karl Pearson "On lines and planes of closest fit," *Phil. Mag.* 6 Ser. Vol. 2, 1901, and L. J. Reed, "Fitting Straight Lines," *Metron*, Vol. 1, 1921.

[14] C. F. Roos, "A General Invariant Criterion of Fit for Lines, Planes, and Functions Expandable in Series, Where all Variates are Subject to Error," to be published in *Metron*. Fundamental contributions using different approaches have also been made in recent years. See, for instance, Harold Hotelling, "Analysis of a Complex of Statistical Variables into Principal Components," *Journal of Educational Psychology*, Sept. and Oct., 1933; Ragnar Frisch, *Statistical Confluence Analysis by means of Complete Regression Systems*, Oslo, 1934; L. L. Thurstone, *Vectors of Mind*, 1935; and M. A. Girshick, "Principal Components," *Jour. Amer. Stat. Ass'n.*, Vol. 31, (1936), pp. 519–528.

+.5195 and that between X and Z is +.7764. Both correlations are positive so that we may take

$$-\tan\alpha = -k = \frac{\delta(y/\sigma_y)}{\delta(x/\sigma_x)} = \frac{(.5195)^2}{(.7764)^2} = +.44,$$

TABLE III
COST AND PRICE IN THE COTTON GOODS INDUSTRY

Date	Wholesale Prices of Cotton Goods Semi-Annual Avj.	Farm Prices of Cotton Goods Semi-Annual Avj.	Av. Hourly Earnings of Labor Semi-Annual Avj.
1928			
Jan.	153.7	100.1	42.5
July	151.0	100.5	43.2
1929			
Jan.	149.7	99.3	42.8
July	140.0	97.7	42.2
1930			
Jan.	119.3	90.7	41.9
July	84.2	78.8	41.2
1931			
Jan.	74.2	71.0	39.1
July	51.3	61.0	38.0
1932			
Jan.	44.5	64.7	36.5
July	48.3	53.6	33.3
1933			
Jan.	53.3	54.2	32.9
July	64.7	87.4	43.1
1934			
Jan.	81.2	87.3	45.9
July	96.5	86.0	47.0
1935			
Jan.	94.8	83.1	46.8
July	86.7	84.1	46.1

We, therefore, transform X and Y so that

$$x = \frac{X - (\text{mean of } X)}{\sigma_x},$$

$$y = \frac{Y - (\text{mean of } Y)}{\sigma_y},$$

and substitute directly in the equations of condition. If all observations are given the same weight, then $p_i = 1$ and consequently the line of best fit is

$$-\left(\sum y_i^2 + .44\sum x_iy_i\right)x + \left(\sum x_iy_i + .44\sum x_i^2\right)y = 0,$$

or, more specifically,

$$-1.3953x + 1.3384y = 0.$$

If one of the correlation coefficients r_{yz} or r_{xz} had been negative, the number k would have been positive. Similar considerations enable us to determine completely the signs of the direction cosines in the more general case of fitting a hyperplane.

II. THE ANALYSIS OF VARIANCE OF TIME SERIES

7. *The Variance of Trend.*—The method of correlation is, of course, not the only one to be used in analyzing time series. Another statistical problem presented by time series, in general, and economic time series, in particular, is essentially that of untangling the effects of the elements of secular trend which we shall designate by T, the quasi-harmonic elements designated by H, and the erratic element designated by E.

In order to analyze the contention of one school of economic thought that economic time series are of the nature of accumulated random series and hence essentially unpredictable a set of 1,200 tosses of a coin was accumulated and compared by Professor Harold T. Davis[15] with the 1,200 monthly items in the series of rail stock prices from 1831–1930. The penny tossing series was accumulated by assuming a unit rise for each head that appeared and a unit fall for each tail. The essential difference between the two series was found in the fact that the standard deviation (with trend removed) for the penny tossing series was given by the formula

$$\sigma = .707\sqrt{n},$$

(where n is the number of intervals from the origin) that is to say, it increased with the length of the series, whereas the standard deviation of the rail stock series quickly reached a maximum value and was essentially constant with respect to the length of the series.

A second problem studied was concerned with an investigation of the trend of rail stock prices by months for 100 years. A trend of 20 years, 1830–1850, was first fitted to the data and then extrapolated four years as a forecast. A period of four years was then deleted from the beginning of the series and four years of actual data from 1850–1854 were added to determine a new trend. This process was continued throughout the whole 100 years of the series, twenty-one forecasts of secular trend thus being secured. It was found that the extrapolated trends showed a wide divergence from one 4 year period to another.

[15] Harold T. Davis, "Significance of Analyses of Variance of Time Series," a lecture delivered at the 1936 sessions of the Cowles Commission Seminars.

The question thus proposed was to account for this wide divergence in the forecasted trends. In order to do this Davis adopted as follows some suggestive ideas set forth in a paper by Henry Schultz.[16]

When a straight line trend, $y=a+bt$, t ranging from $-p$ to $-p$, is used, the standard error of forecast is given by the formula

$$\sigma_f = \epsilon(A + A't^2)^{1/2}$$

$$\epsilon = \sqrt{\frac{n}{n-2}}\,\sigma\sqrt{1-r_{yt}^2}$$

where $n=2p-1$ is the number of items in the series, A and A' are constants depending upon the forecast freedom of the series, and σ is the standard deviation of the original series. If a base period of N units is to be used to forecast the trend of a range of m units, the number of degrees of freedom to be assumed cannot be equal to N but must be assumed equal to N/m. We shall call this ratio $k=N/m$, the *forecast freedom* of the series. Thus for a base series of 20 years and a forecast of 4 years, $k=5$. Hence A and A' must be computed on the assumption that $p=2$.

The hypothesis set forth above was subjected to statistical tests over the entire 100 year rail stock prices using a 20-year base and a 4-year forecast. Since σ_f yields a standard error band, one expects that 68% of the extrapolated trends will lie within the bands. The exact count was 13 inside and 8 outside the bands, a result which fully justified the assumptions since expected values were 14 and 7.

If now on each side of the trend-forecast-bands, one constructs bands equal in width to the standard error, σ, of the original series, then it is to be expected that approximately 46% of the actual items of the time series in the 100 years of forecast will lie within these outer bands. This expectation was justified by the experiment.

It is evident that this method gives a precise range for the forecasting of time series under the assumption that the total variance is compounded from the variance due to linear trend and the variance due to an erratic element.

8. *Significance Tests for Periodogram Analyses.*—For many years attempts have been made to define mathematically periods in economic cycles. Best known of the methods developed is the Schuster technique which is described in detail in Schuster's original papers[17] and in various treatises on Fourier series and the combination of observations, and lucidly in a recent article by Benjamin Greenstein,[18] who has applied the technique to several economic time series and discussed the merits and faults of the method. Also worthy of mention in this connection is a technique developed by Ragnar Frisch[19] for extracting components

[16] Henry Schultz, "The Standard Error of a Forecast from a Curve," *Journal of the American Statistical Association,* Vol. 25, 1930, pp 139–185.

[17] A. Schuster, "On the Periodicities of Sunspots," *Phil. Trans. of the Royal Soc.* Series A, Vol. 206, 1906.

[18] Benjamin Greenstein, "Periodogram Analysis with Special Application to Business Failures in the United States," *Econometrica,* Vol. 3 (1935) pp. 170–198.

[19] Ragnar Frisch, "Changing Harmonics and Other General Types of Components in Empirical Series," *Skandinavisk Aktuarietidskrift,* (1928), pp. 220–236.

with changing periods. However, since Professor Frisch's method must require the use of a considerable number of the degrees of freedom in a series and no discussion of probabilities is given, his technique must, at present, be regarded as untried. On the other hand, although the Schuster technique is only applicable to the determination of fixed periods, definite probability distributions for it have been discovered by R. A. Fisher.[20]

The related problem of using one or more harmonic elements in a time series to improve the forecast freedom of the series has recently been considered by Professor Harold T. Davis.[21] To quote from Professor Davis' writings:

> It should be recognized first that the probabilities involved in determining the structure of a time series must be regarded for the present as belonging to the category of inverse rather than direct probabilities, since the structure is discovered by observation and does not rest upon any secure *a priori* basis. Thus, having taken numbers out of a hat, that is to say, having examined the historical series, one proceeds to guess the distribution of a second sample before it is drawn.
>
> As an example, let us consider the probability that there exists a 40-month cycle in industrial stock prices. This harmonic element is clearly revealed with a large significance for the period from 1897–1913. It again appears as the important harmonic in stock averages from 1914 to 1924 but is effaced in the following bull market. On the probabilities involved here one would expect to find the 40-month cycle again appearing prior to 1897. This expectation is justified since the harmonic again appears in these averages from 1872 to 1897. When rail stock prices are employed, the 40-month harmonic appears in the data prior to the Civil War, but it is again effaced in the turbulence of the Civil War period.

At this point, it may be noted that Professor Davis has devised a technique involving the use of Hollerith tabulating machines which enables him to determine all harmonics from 1 to 75 units in period for five series of three hundred items each in approximately one week.

CHARLES F. ROOS

* * * * *

We have thus seen that rapid strides in developing technique for the analysis of time series have been made in recent years; but there remain many perplexing questions to be answered.

Cowles Commission for
Research in Economics,
Colorado Springs, Colo.

[20] R. A. Fisher, "Test for Significance in Harmonic Analysis," *Proc. Royal Soc. of London*, Vol. 125(A), 1929, pp. 54–59.

[21] Harold T. Davis, "Significance Tests for Periodogram Analyses with Applications to Prices of Common Stocks," lecture presented at the 1936 sessions of the Cowles Commission Seminars.

THE CHICAGO MEETINGS OF THE ECONOMETRIC SOCIETY DECEMBER 28–30, 1936

THERE WILL BE American meetings of the Econometric Society as usual at the end of the year with the social science societies. The sessions will be held in Chicago and the program will begin with a joint meeting of the Econometric Society and the American Statistical Association on the morning of December 28, the topic and papers being as follows:

STATISTICS AND ECONOMETRICS

1. "The Evolution of Fundamental Statistical Techniques," Harold Hotelling, President of the Econometric Society.
2. "Some New Indexes of Agricultural Supplies and Carry-over," E. J. Working, University of Illinois.
3. "New Indexes of Stock Prices and Yields and Their Relation to the Theories of Capital and Savings," Charles F. Roos, Cowles Commission for Research in Economics.

Among those who will discuss these papers will be Irving Fisher, Louis Bean, Willard Thorpe, and S. S. Wilks.

Another session will be devoted to studies of demand. In one paper Holbrook Working will discuss three different types of demand for wheat and in particular the elasticity of demand for wheat for human consumption. Also a paper will be given by William R. Pabst of Amherst College, the title being, "Butter and Oleomargarine: An Analysis of Competing Commodities."

Other sessions will be devoted to (1) Income, (2) General Economic Theory, and (3) Statistical Techniques. Among the speakers will be Irving Fisher who will discuss the nature of income and its relation to income taxation, Herbert E. Jones of the Cowles Commission who will discuss the correlation of time series, and Gerhard Tintner who will consider the relation of prices to the trade cycle.

A general session, at which an industrial engineer will discuss the relation of the economist to industry, is also being planned.

Members of the Society wishing to present papers at these meetings should immediately notify the chairman of the program committee, Professor Harold Hotelling, Columbia University, or the Secretary, Professor Charles F. Roos, 301 Mining Exchange Building, Colorado Springs, Colorado.

LIST OF MEMBERS OF THE ECONOMETRIC SOCIETY

As of October 1, 1936.

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301 Mining Exchange Building, Colorado Springs, Colorado, U. S. A.,
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GROWTH AND DISTRIBUTION OF THE MEMBERSHIP OF THE ECONOMETRIC SOCIETY

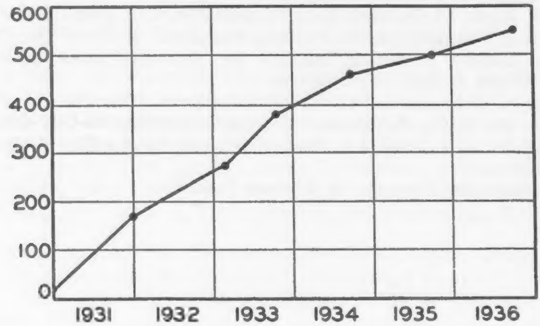

FIGURE 1.—Number of Members of the Econometric Society as of December 29, 1930, January 1, 1932 (these 163 constitute Charter Members), February 11, 1933, and October 1, 1933, 1934, 1935, and 1936.

GEOGRAPHICAL DISTRIBUTION OF ECONOMETRIC SOCIETY MEMBERS

United States	193	Norway	12	Egypt	3
France	52	Austria	12	Mexico	2
Italy	38	China	12	Chile	2
England	35	India	8	Canada	1
Holland	25	Japan	8	Estonia	1
Germany	22	Belgium	8	Finland	1
Poland	17	Bulgaria	4	Algeria	1
Sweden	15	Roumania	4	Scotland	1
Hungary	15	Australia	4	South Africa	1
Czechoslovakia	14	Yugoslavia	4	El Salvador	1
Switzerland	14	Spain	3	Philippine Is.	1
Denmark	13	Argentina	3		
				Total	550

NUMBER OF ECONOMETRIC SOCIETY MEMBERS PER ONE MILLION OF POPULATION, BY COUNTRIES

Norway	4.25	Italy	.91	Roumania	.21
Denmark	3.66	Estonia	.89	Egypt	.21
Switzerland	3.44	El Salvador	.69	Algeria	.15
Holland	3.10	Bulgaria	.66	South Africa	.13
Sweden	2.41	Australia	.60	Japan	.12
Austria	1.78	Poland	.51	Mexico	.12
Hungary	1.73	Chile	.47	Spain	.10
United States	1.52	Germany	.34	Canada	.10
France	1.24	Yugoslavia	.28	Philippine Is.	.07
Belgium	.97	Finland	.27	China	.03
Czechoslovakia	.95	Argentina	.25	India	.03
England	.94	Scotland	.21		

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